SINGULAR LINES IN THE PARAMETER PLANE

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by

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ABSTRACT

The theory of Singular Lines, constant damping ratio-constant undamped natural frequency lines, is derived.

A limitation of the parameter plane method for characteristic polynomials whose coefficients are linear functions of two variable parameters which results in undetermined roots is described. The addition of singular lines to the parameter plane diagram specifies these roots allowing solution for all roots of a given polynomial.

A general method of solving for singular lines is developed and rules for predicting the existence of such lines are stated. A computer program which solves for and graphically displays singular lines in addition to constant zeta, omega, and sigma loci is presented.

Singular lines are considered in terms of dominance and macroscopic root sensitivity. A dominant root line in the parameter plane is illustrated.

Examples which demonstrate the application of singular line theory to linear, multivariable control systems are presented.

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I INTRODUCTION

Prior to 1959 analysis and synthesis of control systems were carried out primarily by root locus and frequency response techniques. [1] Mitrovic's Method [2], which was introduced in 1959, allowed the determination of a system's root locations as two coefficients of the characteristic equation were varied. The method specified the variable coefficients, B_1 and B_0, as the two lowest order terms of the characteristic equation. For example:

$$a_n s^n + a_{n-1} s^{n-1} + \ldots.. + a_2 s^2 + B_1 s + B_0 = 0 \qquad (1\text{-}1)$$

where:

a_k $k = 0, 1, \ldots, n$ are real constants

B_1 and B_0 are real variables

Application of the method resulted in constant zeta, omega, and sigma curves plotted on the $B_0 - B_1$ plane specifying the roots of equation 1-1 for any choice of ζ and ω_n. The Coefficient Plane Method includes Mitrovic's original work and extension of his method to include variation of any two of the coefficients of the system characteristic equation.

In 1964, Siljak [3] extended Mitrovic's work to control systems in which two adjustable parameters appeared linearly in the coefficients of the characteristic equation. In the linear system case, Siljak considered the characteristic equation:

$$f(s) = \sum_{k=0}^{n} a_k s^k = 0 \qquad (1\text{-}2)$$

where: $a_k = b_k\alpha + c_k\beta + d_k$ (1-3)

and: b_k, c_k, d_k are real constants

α, β are real, variable parameters

This method produced constant zeta, omega, and sigma curves plotted on the $\alpha\ \beta$ plane specifying the roots of equation (1-2) for any choice of α and β. The Parameter Plane Method, as Siljak named it, is a simple procedure for factoring characteristic polynomials and displaying the results in a parameter plane diagram.

The Parameter Plane Method was extended in 1965 to a general case in which coefficients of the characteristic equation are a nonlinear combination of two adjustable system parameters. [4], [5], [6] Specifically, the product case where the coefficients, a_k, of equation (1-2) were:

$$a_k = b_k\alpha + c_k\beta + h_k\alpha\beta + d_k \qquad (1\text{-}4)$$

where:

b_k, c_k, h_k and d_k are real constants

α, β are real variable parameters

A further extension of the method to include characteristic equation coefficients of quadratic form is limited to the specific case of third order systems. [7]

The Parameter Plane Method was conceived as an approach to the analysis and synthesis of feedback control systems. It allows the designer to obtain information about system relative stability and the effect of parameter adjustments on stability. By adjusting pole-zero

locations in the system transfer function the designer maintains control over both transient and frequency responses. [8] The method is particularly applicable to multiparameter, multiloop control systems with more than one adjustable parameter since it is based on the study of the system characteristic equation written in a general form. In general the parameter plane method may be applied effectively to any engineering problem in which it is necessary to determine how variations of parameters in the characteristic equation effect root locations.

This section will review Siljak's derivation of the parameter plane method for the linear case [3], discuss the use of the parameter plane for system stability analysis, and illustrate the primary rules for mapping points and curves from the s-plane to the α β plane.

Parameter Plane Equations

Consider the characteristic equation:

$$f(s) = \sum_{k=o}^{n} a_k s^k = 0 \tag{2-1}$$

where:

the coefficients a_k (k=o,1, ..., n) are real and s is the complex frequency variable:

$$s = \sigma + j\omega = -\zeta \omega_n + j\omega_n \sqrt{1-\zeta^2} \tag{2-2}$$

where ω_n is the undamped natural frequency and ζ is the damping ratio

By forming powers of s it can be shown [3] that:

$$s^k = \omega_n^k [T_k(-\zeta) + j \sqrt{1-\zeta^2} U_k(-\zeta)] \tag{2-3}$$

where:

$$T_k(-\zeta) = (-1)^k T_k(\zeta) \tag{2-4}$$

$$U_k(-\zeta) + (-1)^{k+1} U_k(\zeta)$$

The U_k and T_k are Chebyshev functions of the first and second kind respectively. They are given by the recursion formulae [3]:

$$T_{k+1}(\zeta) - 2\,T_k(\zeta) + T_{k-1}(\zeta) = 0$$

$$U_{k+1}(\zeta) - 2\,U_k(\zeta) + U_{k-1}(\zeta) = 0 \qquad (2\text{-}5)$$

where:

$$T_0(\zeta) = 1 \qquad U_0(\zeta) = 0$$

$$T_1(\zeta) = \zeta \qquad U_1(\zeta) = -1$$

Substituting equation (2-3) in equation (2-2) and equating the real and imaginary parts of the resulting equation to zero independently gives:

$$\sum_{k=0}^{n} a_k \omega_n{}^k\, T_k(-\zeta) = 0$$

$$\sum_{k=0}^{n} a_k \omega_n{}^k\, U_k(-\zeta) = 0 \qquad (2\text{-}6)$$

From equations (2-5) we obtain the interrelation equation:

$$T_k(\zeta) = \zeta\, U_k(\zeta) - U_{k-1}(\zeta) \qquad (2\text{-}7)$$

Substitution of equations (2-7) and (2-4) in equations (2-6) produces, after simplification, equations in one Chebyshev function:

$$\sum_{k=0}^{n} (-1)^k a_k \omega_n{}^k\, U_{k-1}(\zeta) = 0$$

$$\sum_{k=0}^{n} (-1)^k a_k \omega_n{}^k\, U_k(\zeta) = 0 \qquad (2\text{-}8)$$

Assume, as Siljak did, that the coefficients a_k of the characteristic equations are linear combinations of parameters α and β of the form:

$$a_k = b_k\alpha + c_k\beta + d_k \qquad (2\text{-}9)$$

where:

b_k, c_k and d_k are real constants

α and β are variable

Substitution of equation (2-9) in equation (2-8) yields:

$$\begin{aligned} B_1\alpha + C_1\beta &= -D_1 \\ B_2\alpha + C_2\beta &= -D_2 \end{aligned} \qquad (2\text{-}10)$$

where:

$$B_1 = \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_{k-1}(\zeta) \qquad B_2 = \sum_{k=0}^{n} (-1)^k b_k \omega_n^k U_k(\zeta)$$

$$C_1 = \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_{k-1}(\zeta) \qquad C_2 = \sum_{k=0}^{n} (-1)^k c_k \omega_n^k U_k(\zeta) \qquad (2\text{-}11)$$

$$D_1 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k U_{k-1}(\zeta) \qquad D_2 = \sum_{k=0}^{n} (-1)^k d_k \omega_n^k U_k(\zeta)$$

The functional dependence of B_1, B_2, C_1, C_2, D_1, and D_2 on ζ and ω_n is omitted in equations (2-10) and succeeding equations for simplicity of notation. A Table of the Chebyshev functions of the second kind, $U_k(\zeta)$ is given in Appendix I. The numerical value of $U_k(\zeta)$ for selected ζ is given in Appendix II.

Application of Cramer's Rule for solution of simultaneous linear equations to equation (2-10) yields the desired parameter plane solution equations:

$$\alpha = \frac{C_1D_2 - D_2D_1}{B_1C_2 - B_2C_1} \qquad \beta = \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1} \qquad (2\text{-}12)$$

As defined by Siljak [3] the parameter plane or $\alpha\beta$ plane is a rectangular coordinate plot with α the abscissa and β the ordinate. Equations (2-12) give α and β as functions of ζ and ω_n. Fixing $\zeta = \zeta_i$ and varying ω_n from 0 to infinity in equations (2-12) produces a curve in the $\alpha\beta$ plane, a constant zeta curve. This curve specifies the $\alpha\beta$ pairs which will cause equation (2-1) to have a pair of complex roots with the required damping ratio ζ_i. Similarly, fixing ω_{ni} and varying ζ from -1 to +1 in equations (2-12) produces a curve in the parameter plane, a constant omega curve, specifying the $\alpha\beta$ pairs which will cause equations (2-1) to have a pair of complex roots with the required natural frequency ω_{ni}. Equations (2-12) thus allow mappings of points, excepting real axis points, from the s-plane to the parameter plane.

For real axis points in the s-plane, replace s in equation (2-1) by $s = -\sigma$ to obtain:

$$f(s) = \sum_{k=0}^{n} a_k(-\sigma)^k = 0 \tag{2-13}$$

Now, substitute equation (2-9) for a_k to obtain:

$$\sum_{k=0}^{n} (b_k\alpha + c_k\beta + d_k)\ (-\sigma)^k = 0 \tag{2-14}$$

Simplifying equation (2-14) yields:

$$\alpha B\ (\sigma) + \beta C\ (\sigma) + D\ (\sigma) = 0 \tag{2-15}$$

where:

$$B(\sigma) = \sum_{k=0}^{n} (-1)^k\ b_k\sigma^k$$

$$C(\sigma) = \sum_{k=0}^{n} (-1)^k\ \overset{c_k}{d_k}\sigma^k \tag{2-16}$$

$$D(\sigma) = \sum_{k=0}^{n} (-1)^k\ d_k\sigma^k$$

For a given value of σ the functions $B(\sigma)$, $C(\sigma)$, and $D(\sigma)$ are constants and equation (2-15) is the equation of a straight line on the $\alpha\beta$ plane, the locus of $\alpha\beta$ points corresponding to real roots $s = -\sigma$.

Stability Analysis and Mapping

Absolute stability analysis of a linear control system consists of determining the existence of roots of the characteristic equation in the right half of the s-plane. Similarly, relative stability analysis consists of determining the existence of roots within specified areas of the s-plane, generally defined by constant ζ or constant ω_n loci. Mapping of the specified areas from the s-plane to the $\alpha\beta$ plane using equations (2-12) and (2-15) allows a designer to choose or adjust parameter values so that characteristic equation roots lie within required areas thereby ensuring stability requirements are met. The rules and graphical techniques for the mapping process and the properties of the mapping are discussed in detail by Siljak [3] and Thaler [9]. The interpretations of the curve on the parameter plane and determination of stability is not a simple process; the reader is again referred to references [3] and [9] for detailed information.

The major mapping rules will be illustrated in the following example.

Example I:

Consider a linear system whose characteristic equation is the fourth order polynomial:

$$f(s) = 0.04\,s^4 + 0.34\,s^3 + (0.2\alpha + 1.12)s^2 + (0.5\alpha + \beta + 1.7)s + 2\beta + 1 = 0 \qquad (2\text{-}17)$$

From the parameter plane diagram for the polynomial shown in Figure 1 we can make the following observations:

1) An s-plane complex point maps into a single $\alpha\beta$ plane point.

2) An s-plane real axis point maps into a straight line, a constant sigma line, on the $\alpha\beta$ plane. Real roots may be evaluated from the σ lines. For example the point M(2.0,0.83) on the $\sigma = 1.5$ line has a root $s = -1.5$.

3) A constant damping ratio, natural frequency or settling time contour maps into a constant ζ, ω_n or $\zeta\omega_n$ curve on the $\alpha\beta$ plane. Complex roots of the polynomial for specific values of α and β are determined by the ζ and ω_n curves. For example point M(2.0, 0.83) is at the intersection of the $\zeta = 0.5$ and $\omega_n = 3.0$ curves, therefore the polynomial has complex roots at $s = -(0.5)(3.0) \pm j\ 3.0\sqrt{1-(0.5)^2}$

$$= 1.5 \pm j\ 2.6$$

4) Constant sigma lines on the $\alpha\beta$ plane are tangent to the $\zeta = 1$ curve at $\omega_n = 1.5$.

5) An s-plane stable area maps into an area of the $\alpha\beta$ plane in which all the roots of the polynomial have negative real parts, i.e., into a stable root area. For example, consider the area in the left half s-plane bounded by the origin, the radial $\zeta = 0.5$ lines, and the infinity of the s-plane. Setting $s = 0$ in equation (2.17) yields $\beta = -0.5$ as the $\sigma = 0$ boundary in the $\alpha\beta$ plane. The $\zeta = 0.5$ radial lines map into the $\zeta = 0.5$ contour of Figure 1. The enclosed area on the plane of Figure 1 is, therefore, the desired stable root area.

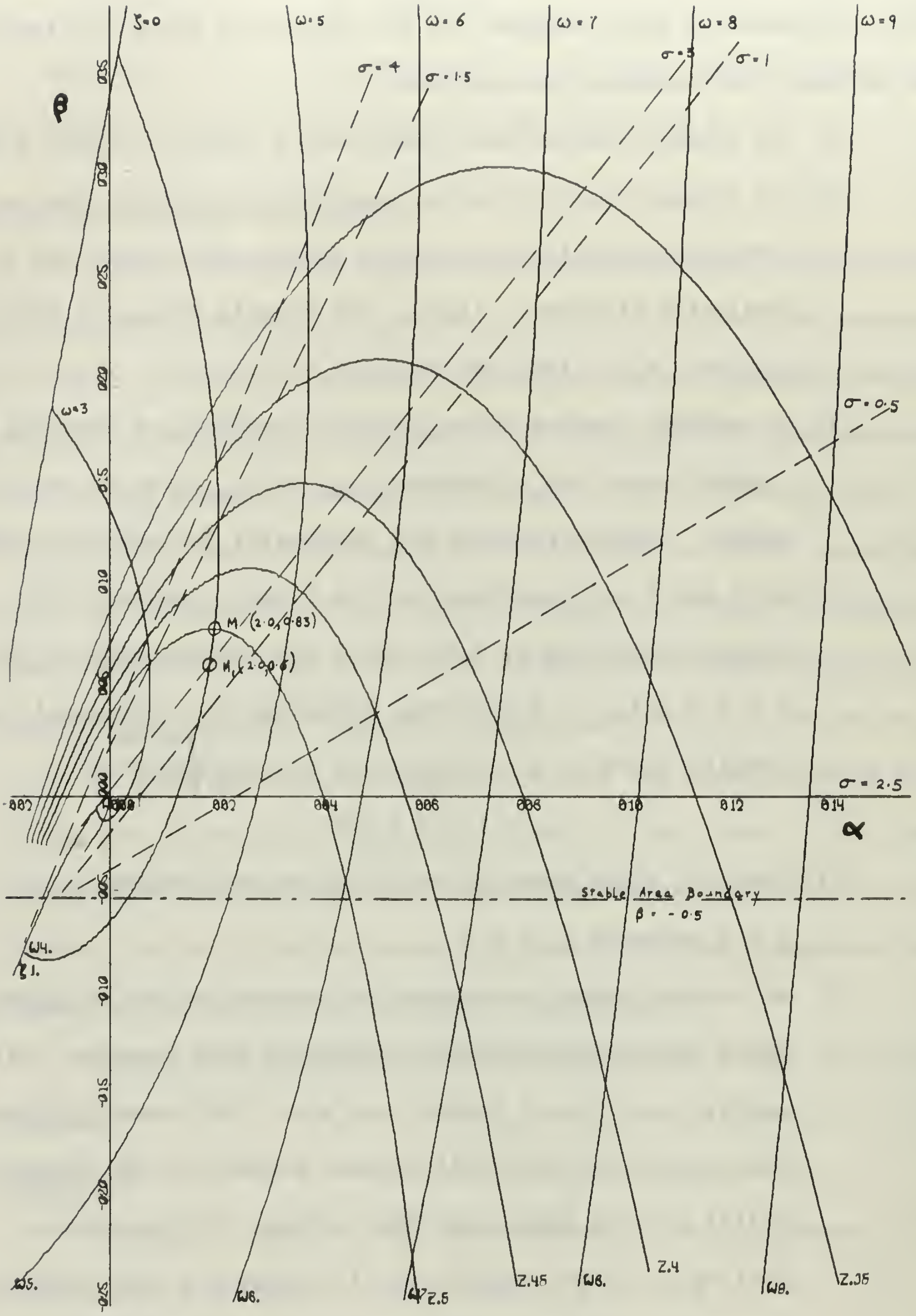


FIGURE 1.

PARAMETER PLANE EXAMPLE
4TH. ORDER POLYNOMIAL

III BASIS OF THIS STUDY

This section introduces a heretofore unknown limitation of the parameter plane method which results in undetermined roots, illustrates the limitation with a specific example, describes initial investigation, and then formulates a hypothesis as to the cause of undetermined roots.

Sections IV and V develop the mathematical theory, solution method, and computer implementation extending the parameter plane method to ensure determination of all roots of a given polynomial. Sections VI and VII consider this extension of theory in terms of dominance and root sensitivity. Section VIII contains examples of analysis of specific linear control systems which indicate the applicability and potential of the theory developed. Section IX concludes this study with comments on work completed and recommendations for further investigation of parameter plane theory and methods.

Limitation of Parameter Plane Theory--Undetermined Roots

The parameter plane method derived by Siljak [3] and the computer program, PARAM A, written by R. M. Nutting [8] purportedly solved the problem of determining all roots of a given polynomial, characteristic equation, in terms of variable parameters α and β displaying the results on the parameter plane as constant ζ, ω_n, and $\zeta\omega_n$ curves and constant σ lines. In most cases the existing parameter plane method does, in fact, solve for all roots of a given polynomial, and in all cases such information as it does produce is correct. There are, however, situations in which the present parameter plane method does not provide a solution for all roots of a polynomial nor correctly predict the existence of complex roots in specific areas of the $\alpha\beta$ plane. Example II is an illustration of such a situation.

Example II

Consider a linear control system whose characteristic equation is

$$\begin{aligned} &s^6 + 80s^5 + (20\alpha + 1600)s^4 + 840\alpha s^3 + \\ &(1600\alpha + 400\beta)s^2 + 1600\beta s + 1600\beta = 0 \end{aligned} \tag{3-1}$$

Solution by the parameter plane method results in the parameter plane diagram shown in Figures 2 and 3. The use of two figures was necessitated by the complexity of the constant ζ curves. Although it is not obvious in Figures 2 and 3 the constant ζ curves all go to the infinity of the parameter plane as ω_n increases. Although the $\zeta = 1$ curve has discontinuities it effectively circles the origin clockwise and then goes to infinity in the second quadrant of the $\alpha\beta$ plane. The computed values of α and β for the constant ζ curves and the use of smaller graph scale for β confirms these statements.

The area of absolute stability, all roots in the left half s-plane, is bounded in the $\alpha\beta$ plane by the $\zeta = 0$ curve, the positive α axis and infinity.

In this example the present parameter plane theory correctly solves for all roots of the polynomial in the complex root area of the $\alpha\beta$ plane in Figure 2. In the real root area bounded by the $\zeta = 1$ curve and the stable area limits the present parameter plane method does not determine all roots of the polynomial; only two or four real roots are determined by constant σ line intersections and no information is presented for the remaining roots. For example, for point M_1 (40, 300) of Figure 2 the roots are

$s_1 = -\ 2.12$ $\quad\quad$ $s_3 = ?$

$s_2 = -37.88$ $\quad\quad$ $s_4 = ?$

$s_5 = ?$

$s_6 = ?$

Similarly, for point M_2 (20, 26.6) the roots are

$s_1 = -\ 2.77$ $\quad\quad$ $s_5 = ?$

$s_2 = -\ 9.77$ $\quad\quad$ $s_6 = ?$

$s_3 = -27.5$

$s_4 = -39.3$

Existence of Undetermined Roots in Parameter Plane Solutions

Further investigation of the situation outlined in Example II yielded the following results. In that part of the parameter plane for which present theory predicted only real roots, specific $\alpha - \beta$ points were found to have either one or two pairs of complex roots. In addition, certain points had the same complex pair in common. For example, for point M_1 (40, 300) of Figure 2 the complex roots are:

$$s_{3/4} = -\ 1.06 \pm j\ 1.83$$

$$s_{5/6} = -18.94 \pm j\ 31.24$$

For points M_2 (20, 26.6) and M_3 (80, 113) the complex roots are

$$s_{5/6} = -\ 0.35 \pm j\ 1.16$$

The results were obtained by substituting specific $\alpha\beta$ pairs in equation (3-1) and solving for the roots of the resulting equation. Moreover, joining points M_2 and M_3 by a straight line and investigating specific points along the line showed that all such points had a common pair of complex roots. Thus a line of constant ζ-constant ω_n was located on the parameter plane.

As previously stated, the existing parameter plane method correctly solves for all roots of a given polynomial in the majority of cases and in all cases the root values which it does determine are correct. It is therefore obvious that existing theory as derived [3] is correct insofar as it goes, but it overlooks or dismisses some special case. Reviewing the derivation, Cf. ante pp. 12-16, it was apparent that the use of Cramer's rule in solving equations (2-10) presumed that the coefficient matrix was non-singular, i.e., that the value of the determinant which formed the denominator in relations (2-12) was non zero, for all cases of interest. This then seemed to be the part of existing theory which could lead to undetermined roots, i.e., incomplete solution of the problem, and this hypothesis formed the basis for the remainder of this study.

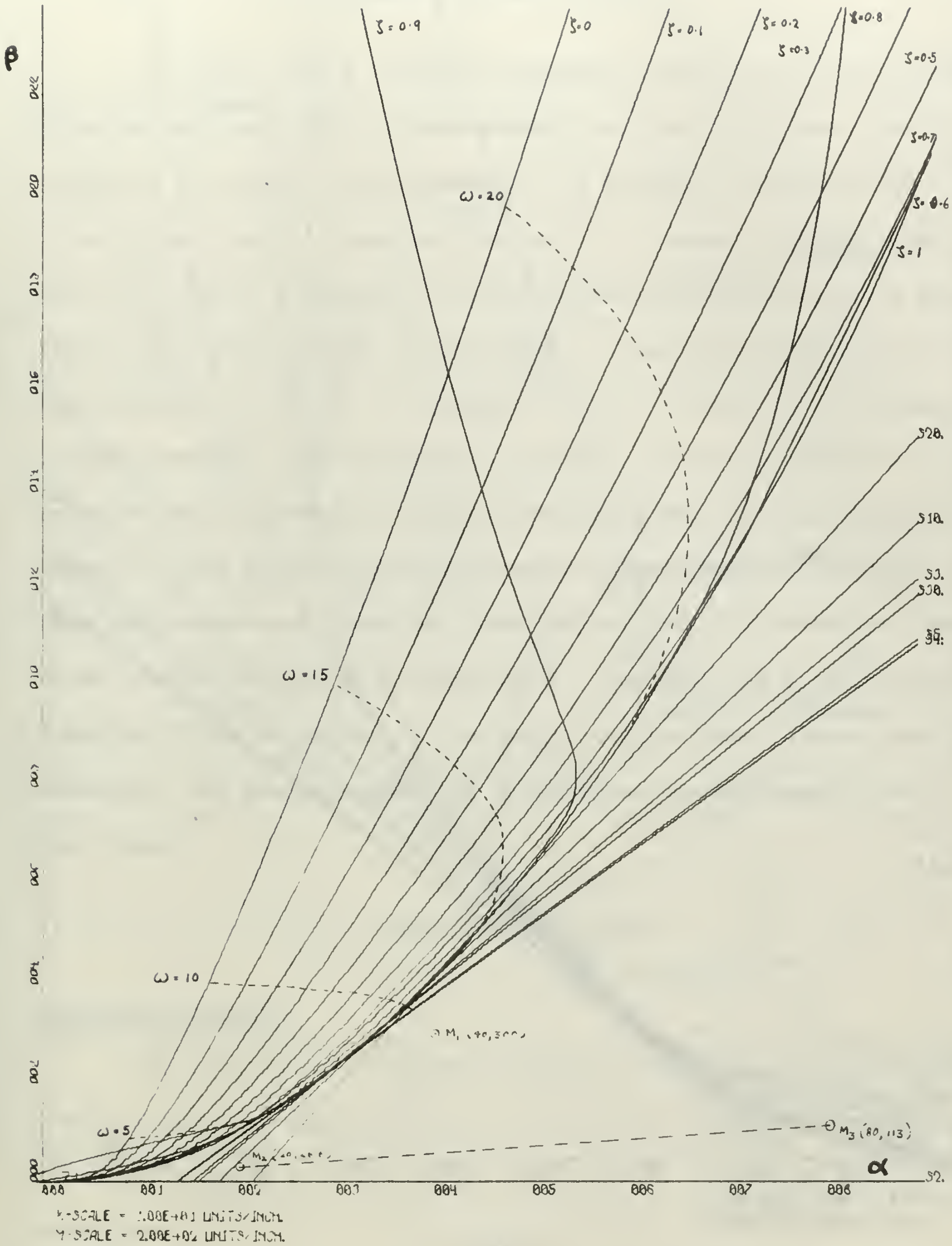


FIGURE 2

PARAMETER PLANE EXAMPLE
6TH. ORDER POLYNOMIAL

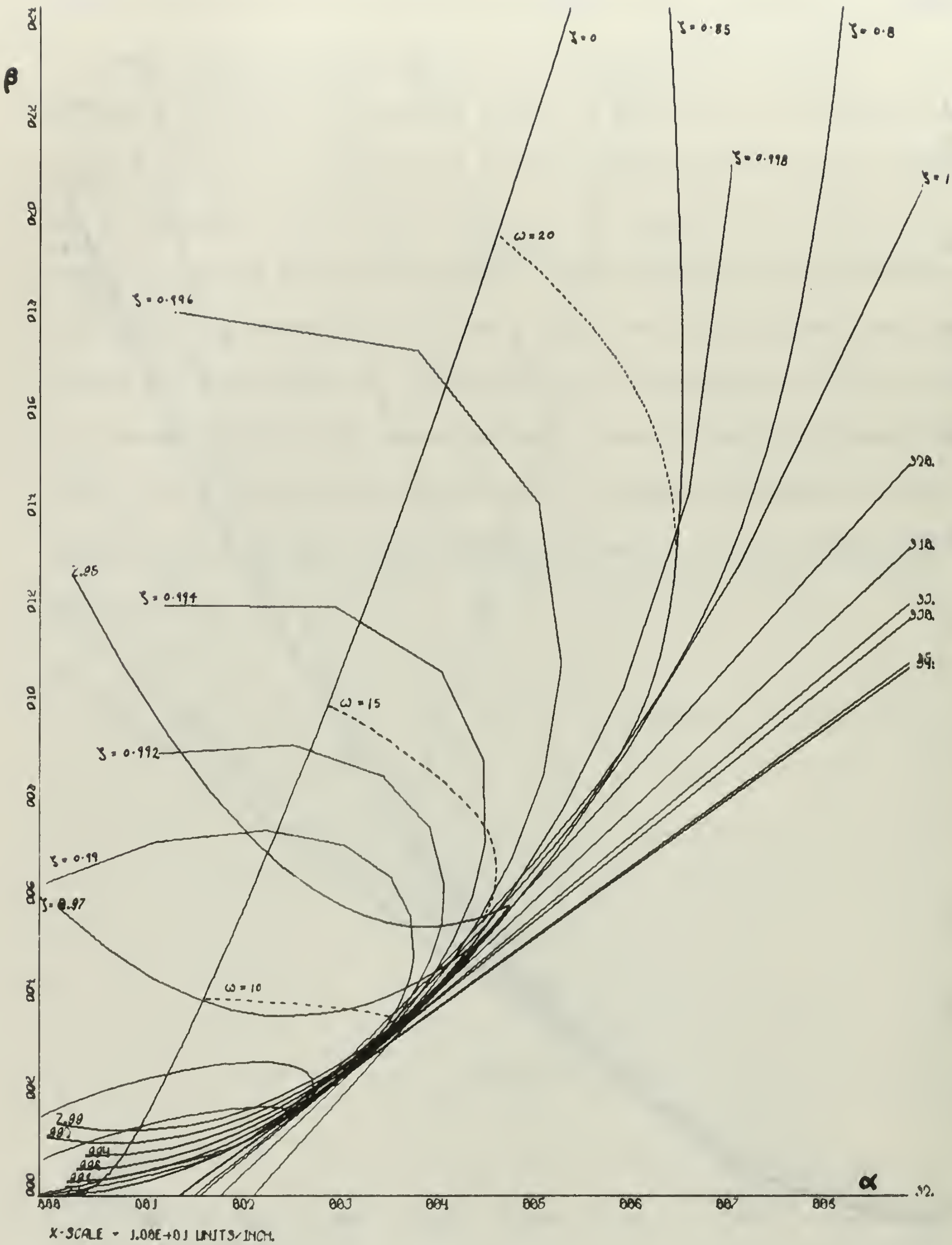


FIGURE 3

PARAMETER PLANE EXAMPLE
6TH. ORDER POLYNOMIAL

IV SINGULAR SOLUTIONS OF LINEAR PARAMETER PLANE PROBLEMS

In Section III a heretofore unknown limitation of the parameter plane method resulting in undetermined roots was described. Example II presented a specific illustration of the problem of undetermined roots. It was shown that at least one constant ζ - constant ω_n line existed and that it provided solutions for some of the roots which existing parameter plane theory could not determine. It was then hypothesized that the assumption that use of Cramer's Rule for solving the simultaneous linear parameter plane equations, equations (2-10), produced the complete solution for all cases of interest excepting that of real roots was incorrect. This section extends parameter plane theory to include the complete solution of equations (2-10) by solving for presently undetermined roots. The theory of constant ζ - constant and ω_n lines defined SINGULAR LINES is derived, a tractable singular line solution method is developed, and general rules for determining the existence of such lines are stated.

I. SINGULAR LINE THEORY

Mathematical Basis

The solution of a system of linear non-homogeneous equations by matrix methods is well known and is contained in any standard textbook of linear algebra [10] or matrix textbook [11]. Consider the system of equations:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ x_n \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ \cdot \\ h_n \end{bmatrix} \qquad (4\text{-}1)$$

or, in compact notation,

$$AX = H \qquad (4\text{-}2)$$

By way of review the following definitions and statements are given without amplification or proof:

1. $A = [a_{ij}]$ is the coefficient matrix.
2. [AH] is the augmented matrix.
3. A system which has a solution is said to be consistent. A consistent system has just one solution, a unique solution, or infinitely many solutions.
4. A system which has no solution is said to be inconsistent. The problem leading to such a system of equations is indeterminate.
5. In the consistent case, the coefficient matrix A and the augmented matrix [AH] have the same rank. In the inconsistent case, they have different ranks.
6. When one or more equations of a system can be derived from another by multiplication of all of their terms by a constant, the equations are dependent or equivalent.
7. The matrix A is called non-singular if its rank $r = n$, that is, if $|A| \neq 0$. Otherwise, A is called singular.
8. The system of equations has a unique solution provided the common rank of matrices A and [AH] is equal to n, the number of variables, that is provided $|A| \neq 0$. Thus the system has a unique solution if A and [AH] have the same rank and if A is non-singular.
9. In a consistent system of rank $r < n$, a solution can be obtained for r variables in terms of the remaining $n - r$ variables.

Derivation of the Singular Line Solution Method

Consider the linear parameter plane equations, equations (2-10), which comprise a system of linear non-homogeneous equations. Transposing terms and rewriting the equations gives

$$\begin{aligned} B_1\alpha + C_1\beta &= -D_1 \\ B_2\alpha + C_2\beta &= -D_2 \end{aligned} \tag{4-3}$$

The coefficient and augmented matrices for equations (4-3) are

$$A = \begin{bmatrix} B_1 & C_1 \\ B_2 & C_2 \end{bmatrix} \tag{4-4}$$

$$AH = \begin{bmatrix} B_1 & C_1 & -D_1 \\ B_2 & C_2 & -D_2 \end{bmatrix} \tag{4-5}$$

The present parameter plane method solves equations (4-3) for those cases in which the common rank of A and [AH] is two as follows:

$$\begin{aligned} \alpha &= \frac{\begin{bmatrix} -D_1 & D_1 \\ -D_2 & C_2 \end{bmatrix}}{\begin{bmatrix} B_1 & C_1 \\ B_2 & C_2 \end{bmatrix}} = \frac{C_1D_2 - C_2D_1}{B_1C_2 - B_2C_1} \\ \beta &= \frac{\begin{bmatrix} B_1 & -D_1 \\ B_2 & -D_2 \end{bmatrix}}{\begin{bmatrix} B_1 & C_1 \\ B_2 & C_2 \end{bmatrix}} = \frac{B_2D_1 - B_1D_2}{B_1C_2 - B_2C_1} \end{aligned} \tag{4-6}$$

There remain, however, the cases in which equations (4-3) are linearly dependent. In these cases matrix A is singular rank one; therefore, the above method of solution is not applicable since the determinant of A, $|A|$, equals zero. To obtain a solution we must first choose a $\zeta - \omega_n$ pair such that matrices A and [AH] have common rank one and then solve one of equations (4-3) for one unknown, β, in terms of the other, α.

For matrices A and [AH] to have common rank one of the following conditions must apply:

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = B_1C_2 - B_2C_1 = 0 \tag{4-7}$$

$$\begin{vmatrix} -D_1 & C_1 \\ -D_2 & C_2 \end{vmatrix} = C_1D_2 - C_2D_1 = 0 \tag{4-8}$$

$$\begin{vmatrix} B_1 & -D_1 \\ B_2 & -D_2 \end{vmatrix} = B_2D_1 - B_1D_2 = 0 \tag{4-9}$$

To choose a $\zeta - \omega_n$ pair which satisfies these conditions, expand the determinant $\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$ in terms of ζ and ω_n, substitute a specific value of $\zeta = \zeta_s$, equate to zero, and solve the resulting polynomial for ω_n. Real roots of the polynomial and the specific value of $\zeta = \zeta_s$ are then substituted in equations (4-8) and (4-9); those values of ω_n which satisfy the equations, ω_{n_s}, are the values for which matrices A and [AH] are both rank one.

Substitution of a $\zeta_s - \omega_{n_s}$ pair in equations (4-3) and solution of the resulting linearly dependent equations for parameter β in terms of parameter α yields

$$\beta = -\frac{D_1}{C_1} - \frac{B_1}{C_1}\alpha \equiv -\frac{D_2}{C_2} - \frac{B_2}{C_2}\alpha \tag{4-10}$$

where:

$$B_1 = \sum_{k=0} (-1)^k b_k \omega_{n_s}{}^k U_{k-1}(\zeta_s) \qquad B_2 = \sum_{k=0} (-1)^k b_k \omega_{n_s}{}^k U_k(\zeta_s)$$

$$C_1 = \sum_{k=0} (-1)^k c_k \omega_{n_s}{}^k U_{k-1}(\zeta_s) \qquad C_2 = \sum_{k=0} (-1)^k c_k \omega_{n_s}{}^k U_k(\zeta_s)$$

$$D_1 = \sum_{k=0} (-1)^k d_k \omega_{n_s}{}^k U_{k-1}(\zeta_s) \qquad D_2 = \sum_{k=0} (-1)^k d_k \omega_{n_s}{}^K U_k(\zeta_s) \tag{4-11}$$

Equation (4-10) is the desired singular line solution. It is the equation of a straight line on the $\alpha\beta$ plane with slope $- B_1/C_1$ and β axis intercept $- D_1/C_1$. All points on the line are maps of the s-plane point (ζ_s, ω_{n_s}) thus it is a constant ζ - constant ω_n line. Substitution of any values of α and β corresponding to the infinity of α-β pairs defined by equation (4-10) in the characteristic equation (2-1) results in s-plane roots:

$$s = - \zeta_s \omega_{n_s} \pm j\omega_{n_s} \sqrt{1 - \zeta_s{}^2}$$

The derivation leading to equation (4-10) is correct, but it is by no means a practical or appealing solution method since the procedure for solving for ζ-ω_n pairs which result in singular lines (ζ_s, ω_{n_s}) is both tedious and time consuming. The presence of Chebyshev functions in equations (2-11) produces polynomials of order n and n-1 in ω_n where n is the order of the given characteristic equation (2-1). Expansion

of equations (4-7), (4-8) and (4-9), which in itself is a tedious process, leads to a polynomial of order 2n-1 in ω_n. For complex systems with higher order characteristic equations the resulting polynomial may be extremely difficult to solve.

In order to derive a more tractable solution method consider a characteristic equation of the form:

$$f(s) = \sum_{k=0}^{n} (b_k\alpha + c_k\beta + d_k)s^k = 0 \qquad (4\text{-}12)$$

Setting n=i for i=2,3; computing B_1, B_2, C_1, and C_2 in equations (2-11); expanding equation (4-7); and grouping terms yields:

For n = 2:

$$B_1C_2-B_2C_1 = [b_0c_1-b_1c_0]\omega_n - [b_0c_2-b_2c_0]2\zeta\omega_n^2 + [b_1c_2-b_2c_1]\omega_n^3 = 0$$

For n = 3:

$$B_1C_2-B_2C_1 = \begin{bmatrix} [b_0c_1-b_1c_0]\omega_n - [b_0c_2-b_2c_0]2\zeta\omega_n^2 + \\ [b_0c_3-b_3c_0](4\zeta^2-1)\omega_n^3 + [b_1c_2-b_2c_1]\omega_n^3 - \\ [b_1c_3-b_3c_1]2\zeta\omega_n^4 + [b_2c_3-b_3c_2]\omega_n^5 \end{bmatrix} = 0$$

Details of the expansion for n=2, 3, 4 and the extension to the general case are contained in Appendix III. The result for the general case is

$$B_1C_2 - B_2C_1 = \begin{bmatrix} [b_0c_1 - b_1c_0]\omega_m + [b_0c_2 - b_2c_0](-1)U_2(\zeta)\omega_m^2 + \cdots\cdots + [b_0c_m - b_mc_0](-1)^{m-1}U_m(\zeta)\omega_m^m \\ + [b_1c_2 - b_2c_1]\omega_m^3 + [b_1c_3 - b_3c_1](-1)U_2(\zeta)\omega_m^4 + \\ \vdots \\ + [b_{i-1}c_i - b_ic_{i-1}]\omega_m^{2i-1} + \cdots\cdots + [b_{i-1}c_{i+j-1} - b_{i+j-1}c_{i-1}](-1)^{i-1}U_j(\zeta)\omega_m^{2i+j-2} \\ \vdots \\ + [b_{m-1}c_m - b_mc_{m-1}]\omega_m^{2m-1} \end{bmatrix} \tag{4-13}$$

or, in compact notation:

$$B_1C_2 - B_2C_1 = \sum_{i,j} ce_{ij} \tag{4-14}$$

where:

$CE = [ce_{ij}]$ is an (nxn) upper triangular matrix whose elements are

$$\begin{cases} [b_{i-1}c_{i+j-1} - b_{i+j-1}c_{i-1}]\,(-1)^{i-1}U_j(\zeta)\omega_n^{2i\ j-2} & i+j < n+2 \\ 0 & i+j \geq n+2 \end{cases} \tag{4-15}$$

Equation (4-14) results in a polynomial in ζ and ω_n of order $2(n-1)$ in ω_n. Substitution of a specific $\zeta = \zeta_s$ and solution for real values of ω_n gives the s-plane coordinates for which the coefficient matrix is singular, equation (4-4). The real values of ω_n and the

specific $\zeta = \zeta_s$ which satisfy equations (4-8) and (4-9) are the coordinates (ζ_s, ω_{n_s}) that ensure matrix A is singular and matrices A and [AH] have rank one. ζ_s and ω_{n_s} are substituted in equation (4-10) which is now a linear equation in two unknowns α and β. The resulting equation defines a line of constant ζ - constant ω_n, a singular line, on the $\alpha\beta$ plane.

The above solution method appears cumbersome when described but, in fact, it proved to be very simple to apply. To illustrate the solution method Examples III and IV are given below. In common with the present parameter plane method, the complexity of computation increases with the order of the given polynomial and computer solution is preferable for higher order-systems.

Example III

Consider the characteristic equation given in Example I:

$$f(s) = 0.04s^4 + 0.34s^3 + (0.2\alpha + 1.12)s^2 + (0.5\alpha + \beta + 1.7)s + 2\beta + 1 = 0 \tag{4-16}$$

For ease of coefficient component identification, display the terms as follows:

k	0	1	2	3	4
b_k	-	0.5	0.2	-	-
c_k	2	1	-	-	-
d_k	1	1.7	1.12	0.34	0.04

Apply equation (4-14) to compute:

$$B_1C_2-B_2C_1 = -\omega_n - 0.4(-2\zeta)\omega_n^{\ 2} - 0.2\omega_n^{\ 3}$$

Simplifying and equating to zero:

$$\omega_n^2 - 4\zeta\omega_n + 5 = 0$$

Choosing $\zeta_s = 0.5$, for example, and solving for the values of ω_n for which matrix A is singular:

$$\omega_n^2 - 2\zeta\omega_n + 5 = 0$$

$$\omega_n = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm j2$$

Since there are no real values of ω_n the matrix is non-singular for all values of ω_n, no constant ζ-constant ω_n line exists, and the present parameter plane method correctly solved for all roots of the given polynomial.

Example IV

Consider the characteristic equation given in Example II:

$$f(s) = s^6 + 80s^5 + (20\alpha + 1600)s^4 + 840\alpha s^3 + (1600\alpha + 400\beta)s^2 + 1600\beta + 1600 = 0 \qquad (4\text{-}17)$$

Display the terms as:

k	0	1	2	3	4	5	6
b_k	-	-	1600	840	20	-	-
c_k	1600	1600	400	-	-	-	-
d_k	-	-	-	-	1600	80	1

Apply equation (4-14) to compute:

$$B_1C_2 - B_2C_1 = \begin{bmatrix} -[0]\,\omega_n & +[1600^2]\,2\zeta\omega_n^2 & -[1600(840)](4\zeta^2-1)\,\omega_n^3 & +[20(1600)](8\zeta^3-4\zeta)\,\omega_n^4 \\ -[1600^2]\,\omega_n^3 & +[1600(840)]\,2\zeta\omega_n^4 & -[1600(20)](4\zeta^2-1)\,\omega_n^5 & \\ -[400(840)]\,\omega_n^5 & +[400(20)]\,2\zeta\,\omega_n^6 & & \end{bmatrix}$$

Simplifying and equating to zero:

$$\zeta\omega_n^4 - (8\zeta^2+19)\omega_n^3 + (16\zeta^3+160\zeta)\omega_n^2 - (336\zeta^2+76)\omega_n + 320\zeta = 0$$

Choosing $\zeta_s = 0.5$, for example, and solving by synthetic division for ω_n gives:

$$\omega_n^4 - 42\omega_n^3 + 164\omega_n^2 - 320\omega_n + 320 = 0$$

$$\omega_n = 2.1115,\ 37.889,\ \text{and a complex pair}$$

Thus the matrix A is singular for $\zeta = 0.5$, $\omega_n = 2.1115$ and for $\zeta = 0.5$, $\omega_n = 37.889$.

Substituting the first pair in equations (4-8) and (4-9) leads to the results 0 = 0 confirming that $(\zeta_s, \omega_{n_s}) = 0.5,\ 2.1115$ is a singular point. From equations (4-11):

$$B_1 = (-1)^2(1600)(2.1115)^2(1)+(-1)^3(840)(2.1115)^3[2(0.5)]+$$
$$(-1)^4(20)(2.1115)^4[4(0.25)-1]$$

$$= -774.06$$

$$C_1 = (-1)^0(1600)(2.1115)^0(-1)+(-1)(1600)(2.1115)(0)+ (-1)^2(400)(2.1115)^2(1)$$

$$= +183.4$$

$$D_1 = (-1)^4(1600)(2.1115)^4[4(0.25)-1]+(-1)^5(80)(2.1115)^5[8(0.125)-2]+ (-1)^6(2.1115)^6[16(.5)^4-12(.5)^3+1]$$

$$= 3268.8$$

Substituting these values in equation (4-10) yields

$$\beta = \frac{3268.8}{183.4} + \frac{774.06}{183.4}\alpha$$

$$= 17.81 + 4.223\alpha$$

which is a line of constant ζ-constant ω_n(.5, 2.1115) whose β axis intercept is 17.81 and whose slope is 4.22. Similarly, the values $\zeta = 0.5$, $\omega_n = 37.889$ satisfy equations (4-8) and (4-9) and define a singular line:

$$\beta = 5740 + 75.777\alpha$$

Mitrovic's Method and the Coefficient Plane Method

Singular line theory must be compatible not only with the parameter plane method but also with its special cases, Mitrovic's method and its extension to the coefficient plane.

The coefficient plane method applies when any two coefficients of the characteristic polynomial, equation (2-1), are variables A_p and A_q where $n \geq p > q \geq 0$. The solution parallels that for the parameter

plane, outlined in section II, up to and including equations (2-8). Substitution of variable coefficients A_p and A_q in equations (2-8) yields:

$$(-1)^p A_p \omega^p U_{p-1}(\zeta) + (-1)^q A_q \omega^q U_{q-1}(\zeta) = -\sum_{\substack{k=0 \\ \neq p,q}}^{n} (-1)^k a_k \omega^k U_{k-1}(\zeta)$$

$$(-1)^p A_p \omega^p U_p(\zeta) + (-1)^q A_q \omega^q U_q(\zeta) = \sum_{\substack{k=0 \\ \neq p,q}}^{n} (-1)^k a_k \omega^k U_k(\zeta) \tag{4-18}$$

Considering these as a system of linear, non-homogeneous equations in two unknowns, A_p and A_q, the coefficient matrix is

$$\begin{bmatrix} (-1)^p \omega^p U_{p-1}(\zeta) & (-1)^q \omega^q U_{q-1}(\zeta) \\ (-1)^p \omega^p U_p(\zeta) & (-1)^q \omega^q U_q(\zeta) \end{bmatrix} \tag{4-19}$$

which simplified is:

$$(-\omega)^{-q} \begin{bmatrix} (-\omega)^{p-q} U_{p-1} & U_{q-1} \\ (-\omega)^{p-q} U_p & U_q \end{bmatrix} \tag{4-20}$$

Considering that $p > q$, inspection of a representative table of Chebyshev functions of the second kind, Appendix II, shows that the determinant of the coefficient matrix, equation (4-9), cannot equal zero. Thus, since the matrix is non-singular except for the trivial case $\omega_n = 0$, equations (4-18) cannot be linearly dependent and the solution for A_p and A_q, if it exists, must be unique.

In order to test the applicability of singular line theory to coefficient plane problems, recast them as parameter plane problems. All possible cases are covered if we consider:

$$A_p \text{ (or } A_q) = f(\alpha)$$
$$A_q \text{ (or } A_p) = f(\beta) \text{ or } f(\alpha,\beta) \tag{4-21}$$

In any case, since only two coefficients of the characteristic equation, (2-1), contain the variable parameters the expansion of the coefficient matrix, equation (4-13), contains one and only one term. The result is that singular line theory always produces the solution for the trivial case, $\omega_{n_s} = 0$, and no singular lines are defined. Singular line theory is, therefore, compatible with the coefficient plane and Mitrovic's Method.

II. CONDITIONS FOR EXISTENCE OF SINGULAR LINES

In order for singular lines to exist the linear, non-homogeneous, parameter plane solution equations, (4-3), must be dependent. The necessary conditions for this are that their coefficient matrix must be singular and their coefficient and augmented matrices must have common rank one.

Ideally an existence theorem which clearly defines the conditions on the characteristic polynomial necessary for singular lines should be stated. This cannot, however, be done since the coefficients and constants in the parameter plane solution equations, equation (4-3), are each involved functions of n+2 quantities where n is the order of the characteristic polynomial, that is, functions of b_k or c_k or d_k and ζ and ω_n. Certain general rules which indicate the likelihood of the existence of singular lines can, however, be formulated. They are

1. The variable parameters α and β must be contained in more than two of the coefficients of the system characteristic equation.

2. Equation (4-13) indicates that for a characteristic equation of order n:

   a. For each value of ζ there is a maximum of $2(n-1)$ values of ω_n which produce singularities, excluding the trivial case of $\omega_n=0$.

   b. For each value of ω_n there is a maximum of $(n-1)$ values of ζ which produce singularities.

   There are an infinity of possible values of ζ for $0 < \zeta \leq 1$, each associated with a maximum of $2(n-1)$ values of ω_n which produce singularities. Thus, if the conditions for singular line existence are met, there is a continuum of such lines spanning the real root section of the parameter plane stable area.

3. Coefficients of α and β in the higher powers of s in the characteristic equation have the most effect on the possible existence of singular points.

4. If in the characteristic polynomial $b_k=c_k=0$ for all $k \geq M$, the maximum possible number of values of ω_n per constant ζ curve which produce singular points is M-1 excluding the trivial case of $\omega_n = 0$.

5. If in the parameter plane diagram the stable area is clearly split into a complex root area in which all the constant ζ contours eventually go to infinity and a real root area which contains only constant σ lines, the possibility of existence of singular lines is high.

V COMPUTER PROGRAMMING OF THE SINGULAR LINE SOLUTION METHOD

Appendix IV is a listing of the digital computer program developed in the course of this study. The program, PARAM S, is comprised of the major parts of PARAM A, a program for solution of the linear parameter plane problem written by R. M. NUTTING [4], and a new sub-program to implement the singular line solution method developed in section IV of this study. Computation and graphing of constant ζ curves, constant ω_n curves, constant σ lines and constant ζ - constant ω_n lines, singular lines, is provided. The many options of quantities computed, curves plotted, and output data printed and the format of necessary input data are explained in the comment sections of Appendix IV. What ?

The computation procedure used in the singular line sub-program for a characteristic polynomial of the form:

$$f(s) = \sum_{k=0}^{n} a_k s^k = 0 \tag{5-1}$$

where:

$$a_k = b_k\alpha + c_k\beta + d_k$$

and for a specified value of damping ratio, ζ, is as follows:

1. Read input data ζ and the coefficients of the characteristic polynomial b_k, c_k, and d_k where $k = 1, 2, \ldots, n+1$.
2. Form the matrix $CE = [ce_{ij}]$ $\quad i=1,2,\ldots,n$

   $\qquad j=1,2,\ldots,n$

   whose elements are

   $$\begin{array}{ll} (b_i c_{i+j} - b_{i+j} c_i) & i+j < n+2 \\ 0 & i+j \geq n+2 \end{array}$$

3. Form the matrix $CF = [cf_{ij}]$ $\quad i=1,2,\ldots,n$

$j=1,2,\ldots,n$

whose elements are

$$cf_{ij} = \begin{cases} (-1)^{i+1}[U_i(\zeta)] & i = j \\ 0 & i \neq j \end{cases}$$

CF is a signed diagonal matrix whose elements are formed from the Chebyshev functions of the second kind for the chosen value of ζ.

4. Generate the AA matrix where:

$$AA = CE \times CF$$

Sum the elements of the AA matrix. The resulting sum is the expansion of $(B_1C_2-B_2C_1)$, equation (4-13). The summation process groups the terms in powers of ω_n and stores the coefficients of ω_n in ascending order.

5. Solve for ω_n using the quadratic formula or subroutine POLYRT as appropriate. POLYRT is a FORTRAN 60 subroutine capable of solving for the roots of a polynomial, order n where $2 < n < 100$, with arbitrary complex coefficients. The subroutine uses Lehmer's method [12] to approximate a root and then improves on it with Newton's method. POLYRT is a relatively slow but accurate subroutine which requires no supervision or initial guesses of root values. For a polynomial with repeated real roots, the results often have erroneous complex parts which are generally less than 2.0×10^{-3}.

6. Test the output of POLYRT retaining only real roots; these, together with the specified value of ζ, define the singular points. If there are no real roots print "Matrix Non Singular", and halt.

7. For each real value of ω_n and the specified ζ generate B_1, B_2, C_1, C_2, D_1, and D_2, equations (2-11); test that the conditions for dependent solution, equations (4-7), (4-8), and (4-9) are satisfied. If $B_1=B_2=C_1=C_2$, print "System of Equations indeterminate. Rank Zero." re loop for next value of ω_n or halt as appropriate. If either (4-8) or (4-9) is not satisfied, print "System of Equations Inconsistent. No Solution." re loop for next value of ω_n or halt as appropriate.

8. If conditions are satisfied computer β for values of α determined by X scale and IYRIGHT input data, equation (4-10), and plot a graph of the singular line. As determined by program options print:

   a. ζ, ω_n, XAXIS intercept, slope, third parameter, and

   b. roots of the characteristic polynomial at selected points on the singular line.

Subroutine POLYRT is again used in solving for the roots. The printout of roots serves both as a check on the correctness of the plotted singular line and as an indication of the movement of other roots as the singular line is traversed.

Example V

Consider the characteristic polynomial:

$$s^4 + (20\alpha+42)s^3 + (20\alpha+400\beta+161)s^2 + (2440\alpha+1600\beta+3280)s + (2360\alpha+1600\beta+1600) = 0 \tag{5-2}$$

The output graph from program PARAM S, the parameter plane diagram, for this polynomial is shown in Figure 4. It is comprised of constant ζ curves, constant ω_n curves, and singular lines, constant ζ- constant ω_n lines, for selected values of ζ and ω_n.

The computed output data associated with each singular line is contained in Appendix V. The two forms of computer printout of singular line information provided by program PARAM S are illustrated. Table III contains the values of ζ, ω_n, XAXIS intercept, and slope for each singular line. Table IV includes the roots of the given polynomial at selected points on a representative singular line.

The constant ζ curves are computed for values of ω_n from .02 to 200. Although it is not obvious in Figure 4, computer data printouts and smaller scale graphs show that the constant ζ contours describe short arcs in the second quadrant of the $\alpha\beta$ plane, jump discontinuously to the fourth quadrant, then follow smooth curves through the fourth and third quadrants prior to proceeding to infinity as shown.

The singular lines in the stable area, effectively bounded by the $\zeta = 0$ contour and the postiive α axis, each define a line of constant ζ - constant ω_n and are coincident with constant σ lines. For example, singular line 43 for which $\zeta_s=0.5$, $\omega_{n_s}=12.723$ is a line of α-β pairs for which the given polynomial has roots:

$$s = -\zeta_s\omega_{n_s} \pm j\omega_{n_s}\sqrt{1-\zeta_s^2}$$

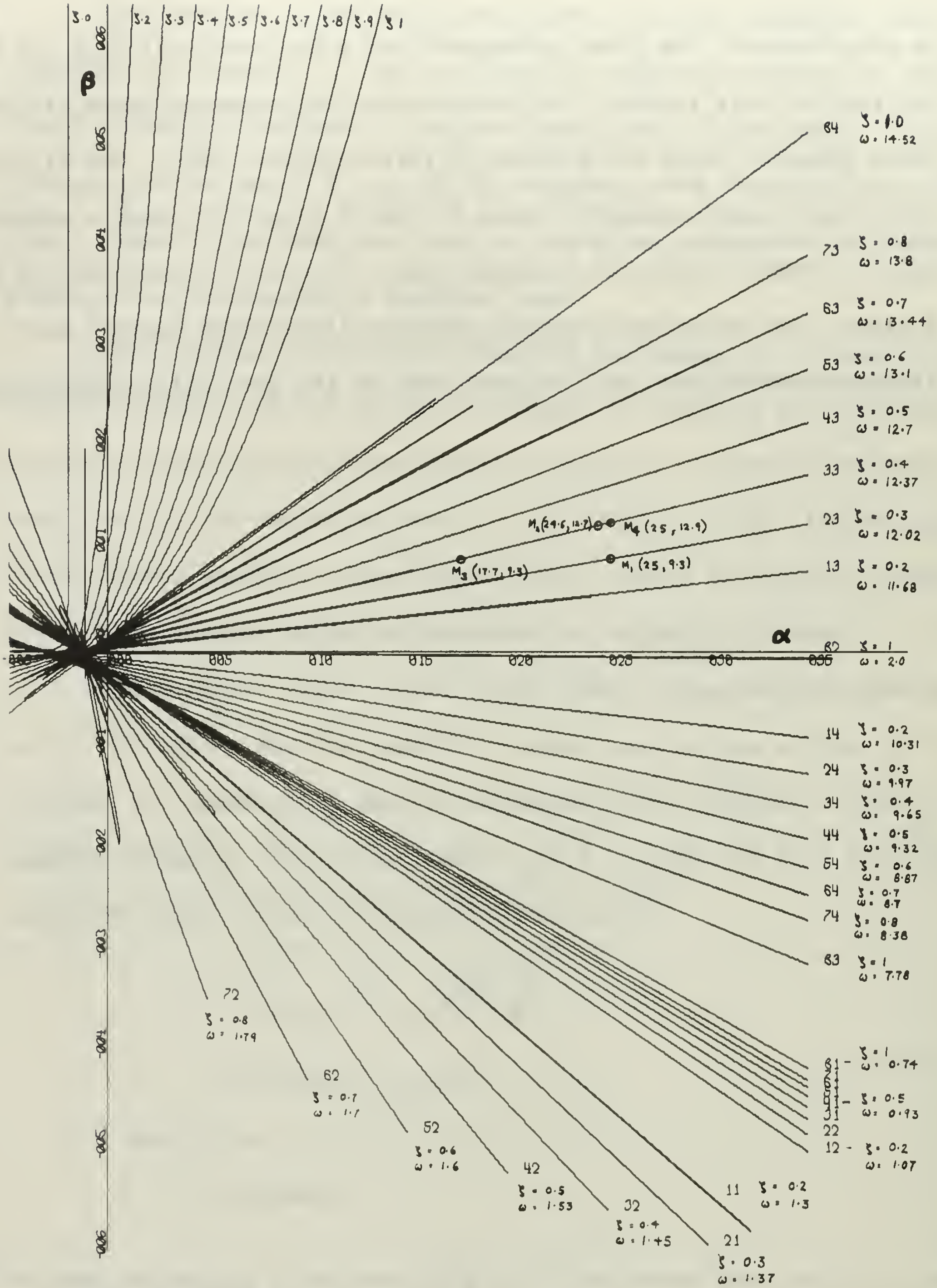
$$= -6.3614 \pm j11.00$$

$$\text{and } s = -\sigma$$

$$= -0.8077$$

As shown in Figure 4 the remaining real root moves from s=71.5 to s=-28.1 as α is increased from -5.0 to 0 and the root continues to move to the left in the s-plane as α is increased further. The remaining singular lines are contained in the unstable region of the $\alpha\beta$ plane; that is, one

or more roots of the given polynomial has a positive real part for any α-β pair in this region. As indicated on the parameter plane diagram, these singular lines are grouped in three distinct sets. Two of the three sets contain singular lines in the $\alpha\beta$ plane for which a specific pair of complex roots, the singular pair, are in the right half of the s-plane. The other set contains singular lines which specify the singular complex pair and one real root in the left half of the s-plane.



K-SCALE = 5.00E+00 UNITS/INCH.
Y-SCALE = 1.00E+01 UNITS/INCH.

FIGURE 4.

SINGULAR LINES ON THE PARAMETER PLANE OF A 4TH. ORDER POLYNOMIAL

VI DOMINANCE OF SINGULAR ROOTS

The concept of dominant mode analysis and design has wide application for linear systems. Most system criteria in the s-plane such as damping ratio, natural undamped frequency, settling time, and peak overshoot have meaning only for a dominant mode system. The no zero, two complex pole system model is accepted as the standard model for s-plane analysis of linear systems. The model characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{6-1}$$

for which the poles are located at:

$$s = -\zeta\omega_n \pm j\omega_n \sqrt{1-\zeta^2} \tag{6-2}$$

When possible a designer attempts to build a system such that the characteristic equation approximates equation (6-1) by choosing one pair of complex roots to satisfy equation (6-2) and forcing the remaining roots to be far to the left in the s-plane or placing zeros to cancel undesired poles. The chosen roots then dominate the transient response. Since the resulting response is essentially that of a second order system, ζ and ω_n may be taken directly from the dominant root location. The peak overshoot, settling time, bandwidth, etc., may then be read directly from a set of standard second order system graphs. In addition, a third order system model with no zero, two complex poles and a real pole in the left half s-plane near the origin may be considered to approximate the standard second order model for analysis purposes.

For systems with higher order characteristic equations the approximate analytical method described in reference [13] may be used for dominant mode design. Essentially, the approximate method consists of

writing the system characteristic equation in terms of its parameters and factoring the characteristic equation to derive two equations, one second order in s. The values of system parameters are then chosen to ensure that the roots of the second order equation are dominant, and the roots of the other equation are far to the left in the complex plane.

Having designed a dominant mode system, if the values of ζ and ω_n pertaining to a singular line on the parameter plane diagram (ζ_s, ω_{n_s}) define s-plane roots $s = -\zeta_s \omega_{n_s} \pm j\omega_{n_s} \sqrt{1-\zeta^2}$ which are equal to or near the chosen dominant roots, then there is a point and possibly many points on the singular line which define values of parameters α and β which ensure dominance. For example, singular line 43 of Figure 4, Example V, for which $\zeta_s = 0.5$ and $\omega_{n_s} = 12.723$, is a dominant root line. Table IV in Appendix V shows that any point on the section of this line in the stable area defines values of parameters α and β for which the characteristic equation (5-2) has the desired dominant roots, a real root near the origin, and a real root far removed in the left half of the s-plane.

When the fortuitous coincidence of chosen dominant roots and singular line fixed roots occurs, as in Example V for singular line 43, a dominant root area may be shown in the $\alpha\beta$ plane. The boundaries of this dominant root area, which are determined by the degree of dominance required, are divergent singular lines. For example singular lines 33 and 53 of Figure 4 bound a dominant root area for which dominancy is between 9.5 and 11.0.

SENSITIVITY ANALYSIS OF SINGULAR LINES

There have been many sensitivity functions defined for the effects of parameter variations on the transfer functions of linear control systems. The root sensitivity functions developed by Kokotovic and Siljak [14] provide the most general solution to the sensitivity problem for small parameter variations in linear control systems, and are most applicable to parameter plane methods. In addition the macroscopic root sensitivity functions defined by F. H. Hollister [5] are applicable for large parameter variations.

Sensitivity Equations

This formulation is an abbreviated composite version of the derivations given by Kokotovic and Siljak [14] and F. H. Hollister [5], and it presents only the salient points. Consider the characteristic polynomial:

$$\sum_{k=0}^{n} a_k s^k = 0 \tag{7-1}$$

where the n coefficients are functions of the system parameters q_r $(r=1,2,\ldots,m)$.

A variation of system parameters q_r will change the coefficients a_k resulting in a change in the n root locations. In order to evaluate the change in the root locations, the real and complex roots are considered separately.

Consider the i^{th} pair of complex roots of equation (7-1) as:

$$r_{i,i+1} = -\zeta_i \omega_{n_i} \pm j\omega_{n_i} \sqrt{1-\zeta_i^{\,2}} \tag{7-2}$$

and the j^{th} real root of equation (7-1) as:

$$r_j = \sigma_j \tag{7-3}$$

Macroscopic sensitivities, the sensitivities of the roots (7-2) and (7-3) to large (finite) changes in parameter q_r, are defined as:

$$S_{i,r}^{\zeta_i} = \frac{\Delta\zeta_i}{\Delta q_n} \qquad q_i=0, i\neq r$$

$$S_{i,r}^{\omega_i} \quad \frac{\Delta\omega_i}{\Delta q_r} \qquad q_i=0, i\neq r \tag{7-4}$$

$$S_{j,r}^{\sigma_j} \quad \frac{\Delta\sigma_i}{\Delta q_r} \qquad q_i=0, i\neq r$$

$S_{i,r}^{\zeta_i}$, macroscopic damping sensitivity, is a measure of the change in the damping ratio ζ_i of the complex root pair r_i due to a finite change in parameter q_r. $S_{i,r}^{\omega_i}$, macroscopic natural frequency sensitivity, is a measure of the change in the undamped natural frequency ω_{n_i} of the complex root pair r_i due to a finite in parameter q_r. $S_{j,r}^{\sigma_j}$, macroscopic real root sensitivity, is a measure of the change in the real root r_j due to a finite change in parameter q_r.

Microscopic sensitivities, the sensitivities of the roots (7-2) and (7-3) to small (infinitesimal) changes in parameter q_r, are defined as:

$$s_{i,r}^{\zeta_i} = \partial\zeta_i/\partial q_r$$

$$s_{i,r}^{\omega_{n_i}} = \partial\omega_{n_i}/\partial q_r \tag{7-5}$$

$$s_{j,r}^{\sigma_j} = \partial\sigma_j/\partial q_r$$

The evaluation of microscopic sensitivities equations, (7-5) is described in detail in reference [14] and is reviewed in reference [5]. In essence it consists of writing the characteristic polynomial as two equations in the Chebyshev functions of the second kind, equations (2-6); computing the partial differentials the functions with respect to the parameter q_i, ζ_i, and ω_i; rearranging the results to get two simultaneous equations in $\partial\omega_i/\partial q_r$ and $\partial\omega_i/\partial q_r$ with coefficients and constants which are summations in U_k, U_{k-1}, U_k^1 and U_{k-1}^1 and a_k; and, solving for the desired sensitivities.

Macroscopic Sensitivity of Singular Lines

The sensitivity of singular lines to finite changes in parameters α and β is determined by applying equations (7-4) using values of α, β, ζ, and ω_n obtained from the parameter plane diagram.

Example VI

Consider the characteristic equation of Example V and the parameter plane diagram, Figure 4. Assume an operating point M_1 (25, 9.3) on singular line 23 for which $\zeta = 0.3$ and $\omega_n = 12.02$. To move to singular line 33 involves an infinity of complex root sensitivities since the singular complex roots are represented by an infinity of points on line 33; however, the following are considered to be of primary interest:

1. Movement of the operating point on a line normal to the terminal singular line. In this case movement to point M_2 (24.6, 12.7) on line 33 for which $\zeta = 0.4$ and $\omega_n = 12.37$. The macroscopic root sensitivities are

$$S_{\alpha}^{\zeta} = \frac{0.1}{24.6-25} = -0.25 \qquad S_{\alpha}^{\omega_n} = \frac{0.35}{24.6-25} = \frac{0.35}{-0.40} = -0.875$$

$$S_{\beta}^{\zeta} = \frac{0.1}{12.7-9.3} = +0.032 \qquad S_{\beta}^{\omega_n} = \frac{0.35}{12.7-9.3} = \frac{0.35}{3.4} = +0.103$$

2. Parameter β fixed. In this case the operating point moves to point M_3 (17.7, 9.3) and the macroscopic root sensitivities are

$$S_{\beta}^{\zeta} = S_{\beta}^{\omega_n} \to \infty$$

$$S_{\alpha}^{\zeta} = \frac{0.1}{17.7-25} = -0.013 \qquad S_{\alpha}^{\omega_n} = \frac{0.35}{-7.3} = -0.048$$

3. Parameter α fixed. The operating point moves to point M_4 (25, 12.9) and the macroscopic root sensitivities are

$$S_{\alpha}^{\zeta} = S_{\alpha}^{\omega_n} \to \infty$$

$$S_{\beta}^{\zeta} = \frac{0.1}{12.9-9.3} = +0.028 \qquad S_{\beta}^{\omega_n} = \frac{0.35}{3.6} = 0.097$$

The sensitivities computed in Example VI show that the frequency of the singular line complex roots is more sensitive than the damping ratio to finite changes in both parameters singly or together. Inspection of Figure 4 shows that this result is true for all singular lines in the stable area since they are uniformly divergent with increasing positive slope for larger damping ratio and smaller natural frequency. In addition, Figure 4 shows that the sensitivity of singular complex roots to parameter variations is decreased as the operating values of α, β, or α and β are increased; that is, as the singular lines diverge sensitivity is decreased.

Complex root sensitivities of the singular lines could be used effectively in self adaptive control systems. For example, consider a system with characteristic equation (5-2) for which the desired operating point is $M_2(29.6,12.7)$ on singular line 33 of Figure 4. Consider that either parameter α or β changes from its design value due to aging, breakdown, or external disturbance resulting in changes of both ζ and ω_n of the complex roots of the system. As shown in Example VI, natural frequency is more sensitive than the damping ratio to finite changes in both α and β. A sensor at the output could measure the system response to a standard input, compare this with a model generated output, and use the difference in transient frequencies to determine the change in natural damping of the dominant roots. A controller could then determine and apply the change in parameter β necessary to drive the operating point back to singular line 33. Since the complex roots are dominant, the real root near the origin is fixed, and the remaining real root is far removed in the left half s-plane for any point on the singular line, the specific point of return to line 33 is not significant. Parameter β is therefore chosen as the correction element since

$$\left(S^{\zeta}_{\beta}\right)_{\Delta\alpha=0} > \left(S^{\zeta}_{\alpha}\right)_{\Delta\beta=0} \text{ and } \left(S^{\omega_n}_{\beta}\right)_{\Delta\alpha=0} > \left(S^{\omega_n}_{\alpha}\right)_{\Delta\beta=0}$$

VIII APPLICATION OF SINGULAR LINE THEORY TO LINEAR CONTROL SYSTEMS

In this section selected linear multivariable control systems are considered in sufficient detail to demonstrate the applicability and potential of singular line theory. The analysis of each system is not complete; discussion is limited primarily to the singular lines on the parameter plane diagram and their interpretation.

The basic system to be analyzed is an inertially stabilized space vehicle. [13] Specifically, the system is comprised of two, cross coupled channels each with reference and disturbance inputs, plant, lead compensator, coupling element, and unity feedback path. Throughout the succeeding examples complete symmetry of plants, compensators, and coupling coefficients is assumed. That is:

$$\begin{aligned} G_1 &= G_2 = G \\ G_{c_1} &= G_{c_2} = G_c \\ a_{11} &= a_{22} \\ a_{12} &= a_{21} \end{aligned} \tag{8-1}$$

Examination of the system flow graph, Figure 5, yields the characteristic equation:

$$\Delta = 1+2a_{11}G_cG+(a_{11}^{\ 2}-a_{12}^{\ 2})(G_cG)^2 \tag{8-2}$$

<u>Example VI</u>

Assume plant and compensator transfer functions:

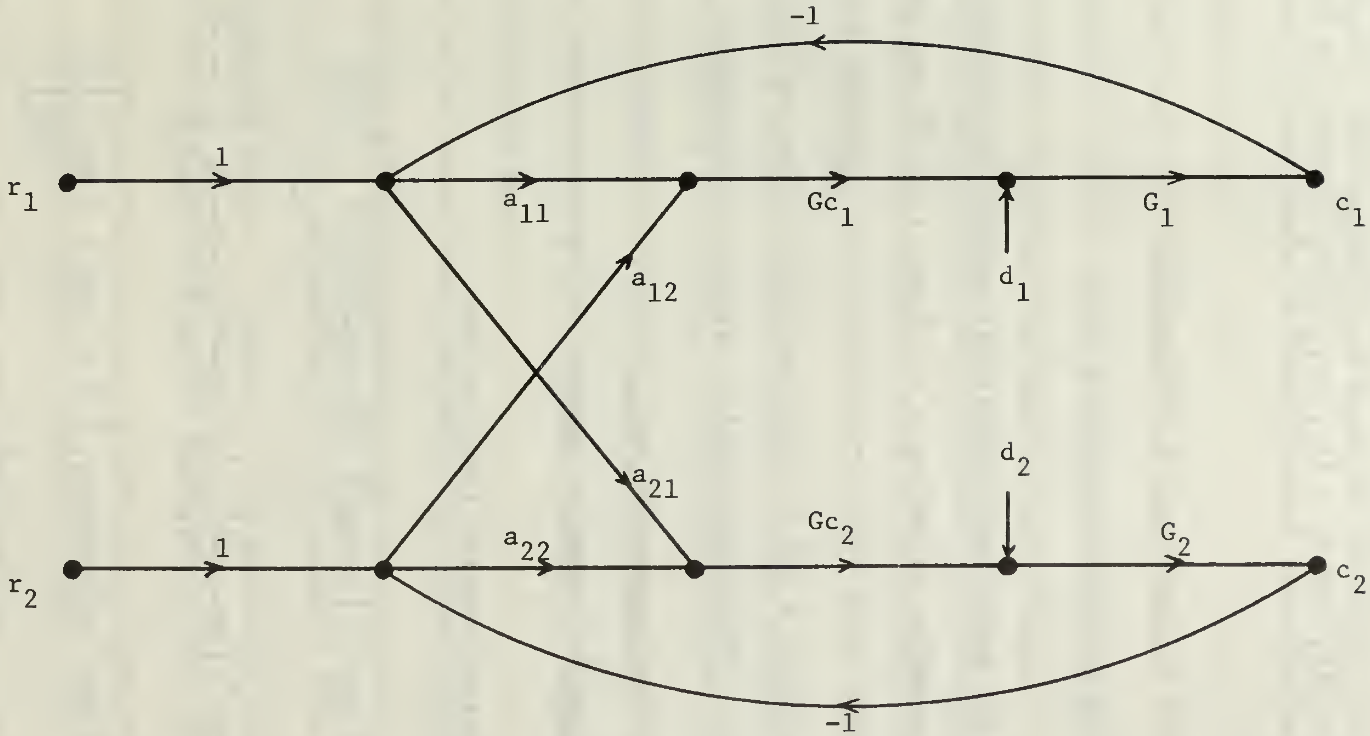


Figure 5

Linear, Multivariable System Flowgraph

$$G = \frac{K}{s^2}$$

$$G_c = K_1 \left(\frac{\tau s+1}{\gamma\tau s+1}\right) \tag{8-3}$$

Substitution of equations (8-3) in equation (8-2) and definition of parameters $q_1 = 2a_{11}KK_1$ and $q_2 = (a_{11}^2 - a_{12}^2)(KK_1)^2$ gives the characteristic equation as:

$$(\gamma\tau)^2 s^6 + 2\gamma\tau s^5 + (1+\gamma\tau^2 q_1)s^4 + (\tau+\gamma\tau)q_1 s^3 +$$
$$(q_1+\tau^2 q_2)s^2 + 2\tau q_2 s + q_2 = 0 \tag{8-4}$$

Using the approximate analytical methods of reference [13] compensator parameter values of $\tau = 0.5$ and $\gamma = 0.05$ are chosen to provide dominant roots with $\zeta = 0.5$ and $\omega_n = 1$. Consider parameters q_1 and q_2 as variables α and β in the parameter plane coefficient equation, equation (2-9). The characteristic equation then becomes

$$\Delta = s^6 + 80s^5 + (1600+20\alpha)s^4 + 840\alpha s^3 +$$
$$(1600\alpha+400\beta)s^2 + 1600\beta s + 1600\beta \quad = \quad 0 \tag{8-5}$$

The system transfer functions are

$$\frac{C_1(s)}{R_1(s)} = \left[s^4 + 42s^3 + \left(80+40\frac{q_2}{q_1}\, s^2\right) + 160\frac{q_2}{q_1}s + \frac{q_2}{q_1}\right] \frac{0.1q_1}{\Delta} \tag{8-6}$$

$$\frac{C_1(s)}{R_2(s)} = \left[s^4+42s^3+\left(80-10\frac{a_{12}}{a_{11}}q_1\right)s^2+40\frac{a_{12}}{a_{11}}q_1 s+40\frac{a_{12}}{a_{11}}q_1\right] \frac{0.1q_1}{\Delta} \tag{8-7}$$

$$\frac{C_1(s)}{D_1(s)} = \left[s^4 + 80s^3 + \left(1600+10q_1\right)s^2 + 420q_1 s + 800\frac{q_1}{K}\right] \frac{K}{\Delta} \tag{8-8}$$

$$\frac{C_1(s)}{D_2(s)} = \left[s^2 + 425s + 80\right] \frac{a_{12}K^2K_1}{2\Delta} \tag{8-9}$$

The numerators of equations (8-6), (8-7), and (8-9) are real functions of one relatively simple parameter formed from the gains and coupling coefficients of the system, i.e., q_2/q_1, $a_{12}q_1/a_{11}$, and $a_{12}K^2K_1/2$. The zeros of these transfer functions can, therefore, be obtained by root locus methods. The numerator of equation (8-6) is a function of parameters K and q_1; the zeros of this transfer function can be specified by two methods:

1. Fix K and vary q_1 to determine the root locus.
2. Designate $q_1 = \alpha$, $\frac{1}{K} = \beta$ and use the quadratic coefficients parameter plane method to obtain a parameter plane diagram.

These transfer functions are included simply to indicate a possible analysis method; no discussion of them will be given since it is outside the scope of this study.

Figure 6 is the complete parameter plane diagram of equation (8-5) for selected values of ζ and ω_n. Associated computer output data is contained in Appendix V, Table V. The constant ζ and constant ω_n curves in Figure 6 were predicted by previous parameter plane theory, but the singular lines were not.

For system specifications requiring an underdamped or critically damped response, previous parameter plane theory restricted values of parameters α and β to the section of the parameter plane bounded by the $\alpha = 0$ contour and the $\zeta = 1$ contour. Thus values of α and β which specified an operating point in the stable real root area were prohibited. This example shows, however, that the singular lines define operating points at which complex roots exist throughout the area previously considered solely a real root area. Each singular line is tangent to an associated constant ζ curve. The value of ω_n on the constant ζ curve at

the point of tangency defines the constant ζ and constant ω_n of the singular line. In addition, in this particular case, the parameter plane diagram shows that:

1. Each singular line is not only a line of constant ζ - constant ω_n for one pair of complex roots, but is also a line for which the value of one real root is constant. Thus the singular lines are coincident with constant σ lines. Specification of the two system parameters q_1 and q_2 such that they define any point on a singular line fixes three of the six roots of equation (8-5). The remaining three roots vary with the position along the singular line. For example, singular line 101 for which $\zeta = 0.9$ and $\omega_n = 3.83$ has fixed roots:

$$s_{1,2} = -3.448 \pm j\,1.670$$
$$s_3 = -33.103$$

2. The intersection of two singular lines defines the values of the system variable parameters q_1 and q_2 necessary to fix all six roots of equation (8-5). For example, operating point $M_1(64.5, 650)$ which is at the intersection of singular lines 72 and 101 results in characteristic equation roots:

$$s_{1/2} = -18.92 \pm j25.22$$
$$s_3 = -21.648 \qquad \text{determined by line 72 and,}$$

$$s_{5/6} = -3.448 \pm j1.67$$
$$s_6 = -33.103 \qquad \text{determined by line 101.}$$

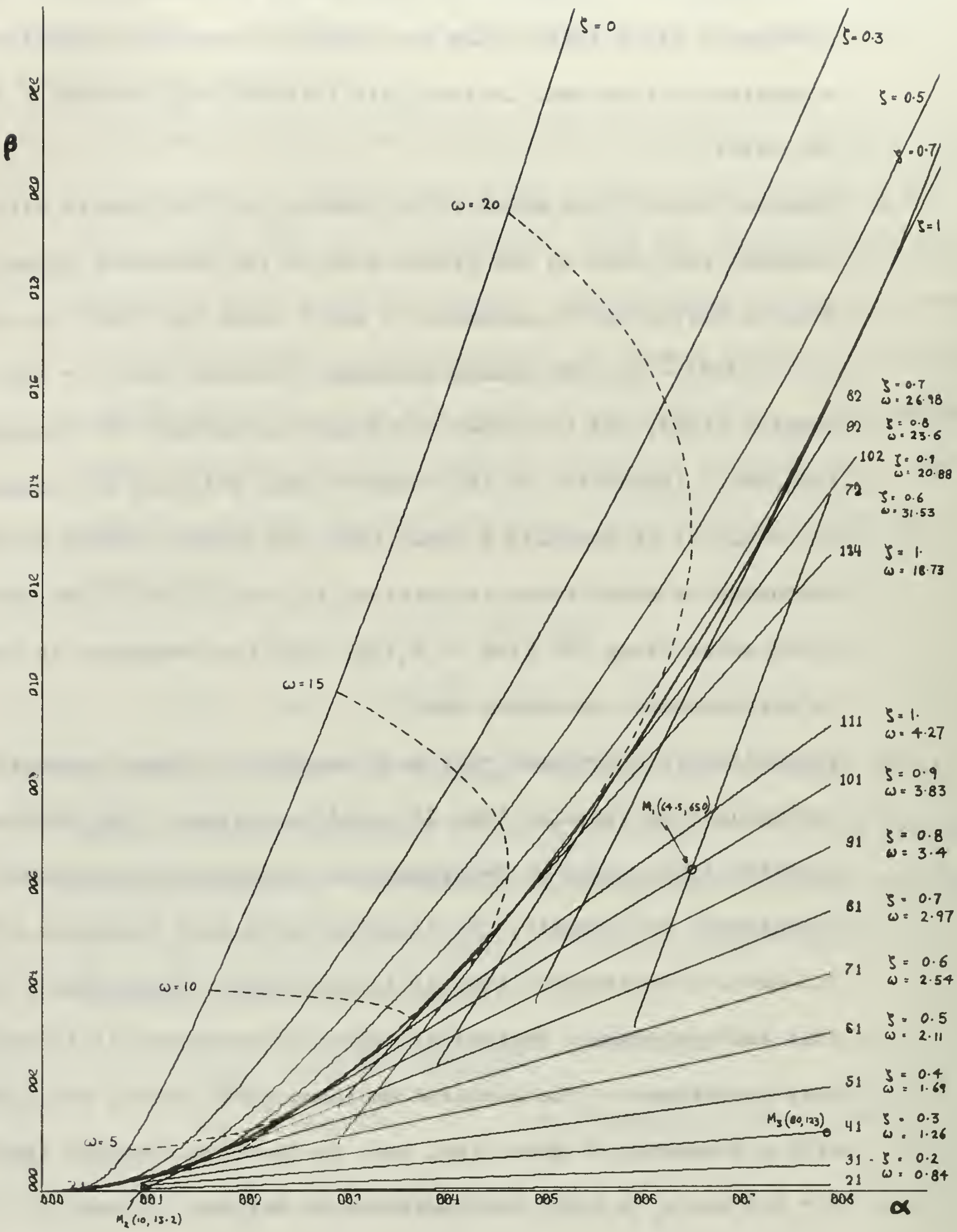
This was to be expected since such an intersection defines unique values of α and β so that the coefficients of equation (8-5) are

all constants. Previous theory predicted the intersection of six constant σ lines rather than two singular lines each coincident with a constant σ line and, in fact, six intersecting constant σ lines do not exist.

3. Singular line 41 for which $\zeta = 0.3$ and $\omega_n = 1.26$ clearly defines a dominant root line in the stable area of the parameter plane. Despite variations in parameter α and β along the line, the roots $s = -0.3793 \pm j1.2062$ remain dominant, the real root $\sigma = -39.24$ remains fixed, and the other roots move to satisfy the characteristic equation. Inspection of the computer data printout for singular line 41, Table VI of Appendix V shows that the singular roots have a dominancy of about three at point $M_2(10,13.2)$, but as the operating point moves along the line to $M_3(80, 123)$ the dominancy is reduced to approximately one point six.

4. Theoretically a dominant root area bounded by chosen singular lines diverging from singular line 31 could be defined. The choice of specific lines would be determined by the degree of dominance acceptable; for example, the singular roots have dominancy of between 2.3 and 1.2 as singular line 31 is traversed. This example shows that the approximate analytical method of reference 13 is at best very approximate. The required dominant roots were $\zeta = 0.5$ and $\omega_n = 1$ with a dominancy of about ten, but, in fact, on singular line 51 with $\zeta = 0.5$ and $\omega_n = 1.055$ the dominancy is between 1.5 and 1.

Figure 7, Sensitivity of Singular Lines to Parameter Variations, is the parameter plane diagram of equation (8-5) for values of ζ between 0.5 and 0.6 and ζ between 0.8 and 0.9. Inspection of the diagram yields the following information. Maximum sensitivities of ζ and ω_n of the



X-SCALE = 1.00E+01 UNITS/INCH.
Y-SCALE = 2.00E+02 UNITS/INCH.

FIGURE 6

SINGULAR LINES ON THE PARAMETER PLANE
OF A MULTIVARIABLE , COUPLED SYSTEM

singular complex roots to finite variations in α, β, and α and β occur at the points of tangency of the singular lines with associated constant ζ contours. That is, points M_1 (8.6,22) for ζ between 0.5 and 0.6 and M_2 (2.175, 119) for ζ between 0.8 and 0.9. As singular lines diverge from these points the macroscopic singular complex root sensitivities, equations (7-4), decrease.

Consider operating point M_3 (36.35, 135.6) on singular line 11 for which $\zeta = 0.5$ and $\omega_n = 2.1115$. Moving the operating point to M_4(36.35, 141.2) on singular line 21, $\zeta = 0.51$, $\omega_n = 2.154$ yields sensitivities:

$$\left(S_{\beta}^{\zeta}\right)_{\Delta\alpha=0} = \frac{0.51-0.50}{141.2-135.6} = \frac{0.01}{5.6} = 0.0018$$

$$\left(S_{\beta}^{\omega_n}\right)_{\Delta\alpha=0} = \frac{2.154-2.1115}{5.6} = \frac{0.039}{5.6} = 0.0070$$

That is, the natural frequency is approximately four times more sensitive to changes in β with α constant than the damping ratio. Moving the operating point to M_5(35.3,135.6) on singular line 21 yields sensitivities:

$$\left(S_{\alpha}^{\zeta}\right)_{\Delta\beta=0} = \frac{0.01}{35.3-36.35} = \frac{0.01}{-1.05} = -0.0095$$

$$\left(S_{\alpha}^{\omega_n}\right)_{\Delta\beta=0} = \frac{0.039}{-1.05} = -0.037$$

That is, the natural frequency is approximately four times more sensitive than the damping ratio to changes in α with β constant. Moreover, a finite change in α has approximately five times the effect of an equal change in β on the natural frequency and damping ratio of the singular lines.

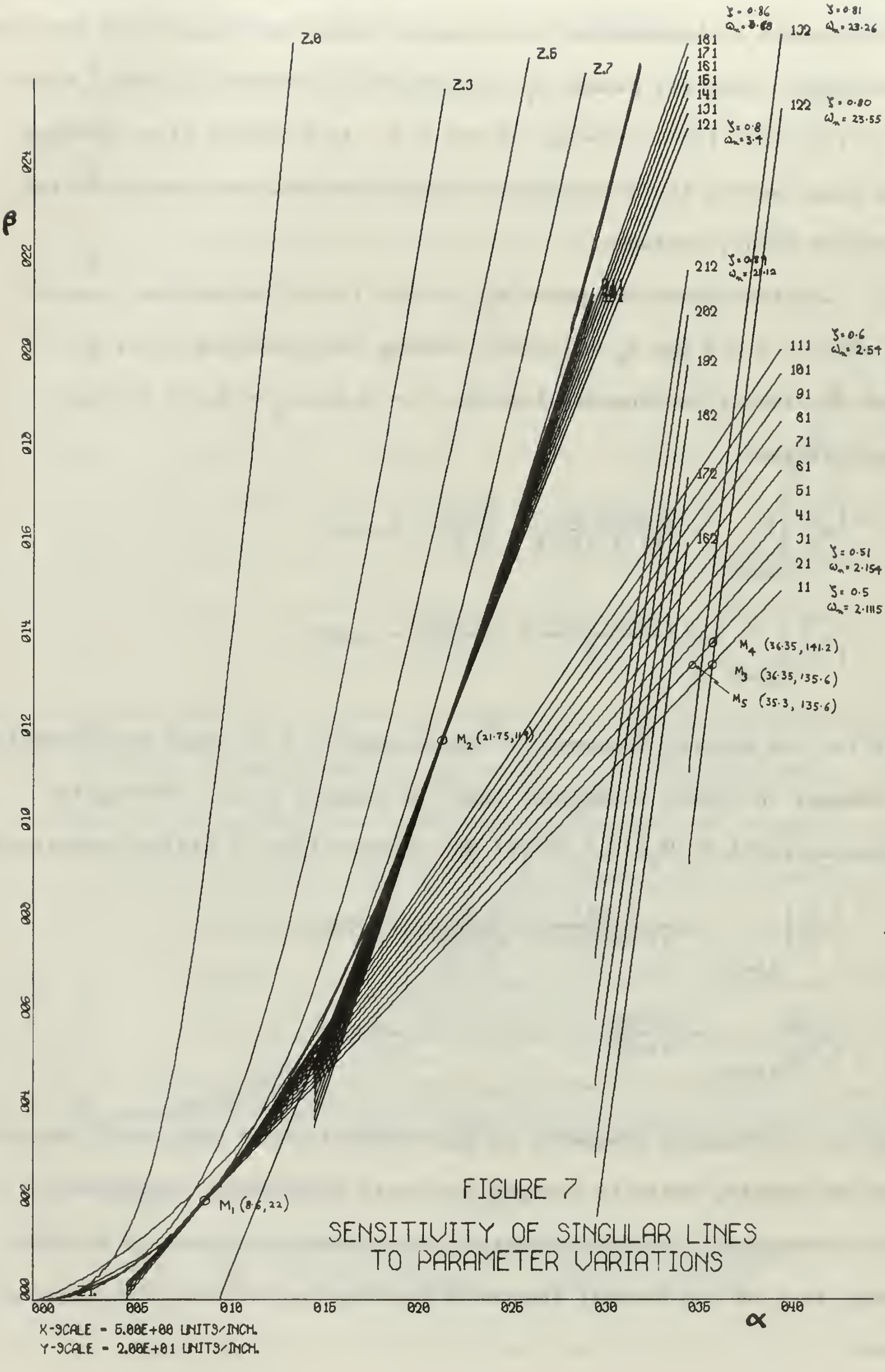


FIGURE 7

SENSITIVITY OF SINGULAR LINES TO PARAMETER VARIATIONS

Example VIII

Consider that the specification for the basic system's response is changed from underdamped with one overshoot to critical damping. Assume the same system structure and transfer functions. The system characteristic equation is equation (8-4). The approximate analytical methods of reference 13 now require compensator parameters $\tau = 0.707$ and $\gamma = 0.05$ to place dominant roots at $\zeta = 0.707$ and $\omega_n = 1$. Considering parameters q_1 and q_2 defined in Example VII as variables α and β, the system characteristic equation is

$$s^6 + 56.56s^5 + (20\alpha+800)s^4 + 593.9\alpha s^3 + (800\alpha+400\beta)s^2 + 1131.5\beta s + 800\beta = 0 \tag{8-10}$$

The parameter plane diagram for equation (8-10) is shown in Figure 8. Computed data for the singular lines is contained in Appendix V, Table VII. The constant ζ curves are similar to those in Figure 6 or Figures 2 and 3 in that they divide the stable region of the parameter plane into two distinct areas. The stable region is bounded by the $\zeta = 0$ contour and the positive α axis. The section of the stable region containing the intersecting constant ζ curves was previously considered the complex root region; existing parameter plane methods correctly solved for all six roots of equation (8-10) in this region. The section of the stable region bounded by the $\zeta = 1$ contour and the positive α axis was considered the stable real root area. Singular lines exist in this area. Previous parameter plane methods determined either four or two real roots for each point in the stable real root area, displaying the reaults as intersections of constant σ lines. Inclusion of singular

lines completes the parameter plane diagram allowing graphical solution for all roots of equation (8-10).

Figure 8, the parameter plane diagram, shows that:

1. The value of constant ω_n for a singular line is determined by the value of ω_n on the constant ζ curve at the point of tangency of the singular line. For example, singular line 51 (ζ=0.5, ω_n=1.4929) is tangent to the ζ=0.5 curve at point M_1(3.95, 3.91) whereas line 52 is tangent to the ζ = 0.5 curve at point M_2(76.4, 1460.).
2. Each singular line is coincident with a constant σ line. For example, line 71 for which ζ=0.7 and ω_n=2.0972 has fixed roots:

$$s_{1,2} = -1.468 \pm j\,1.4977$$

$$s_3 = -25.34$$

3. As in Example VII, the intersection of two singular lines define the value of parameters α and β necessary to fix all roots of equation (8-10).
4. The singular complex roots become dominant at singular line 61, that is for ζ=0.6. The amount of dominancy is increased as the operating point moves to singular lines 71, 81, and 91, i.e., as ζ is increased to 0.9, but the maximum achieved is approximately 1.9.
5. The natural frequencies and damping ratios of singular lines 11 through 101 are more sensitive to finite changes in parameter β than to changes in α. Since these singular lines diverge, effectively from the origin of Figure 8, the singular root sensitivities increase with respect to α but decrease with respect to β as the operating point β value increases with α held constant.

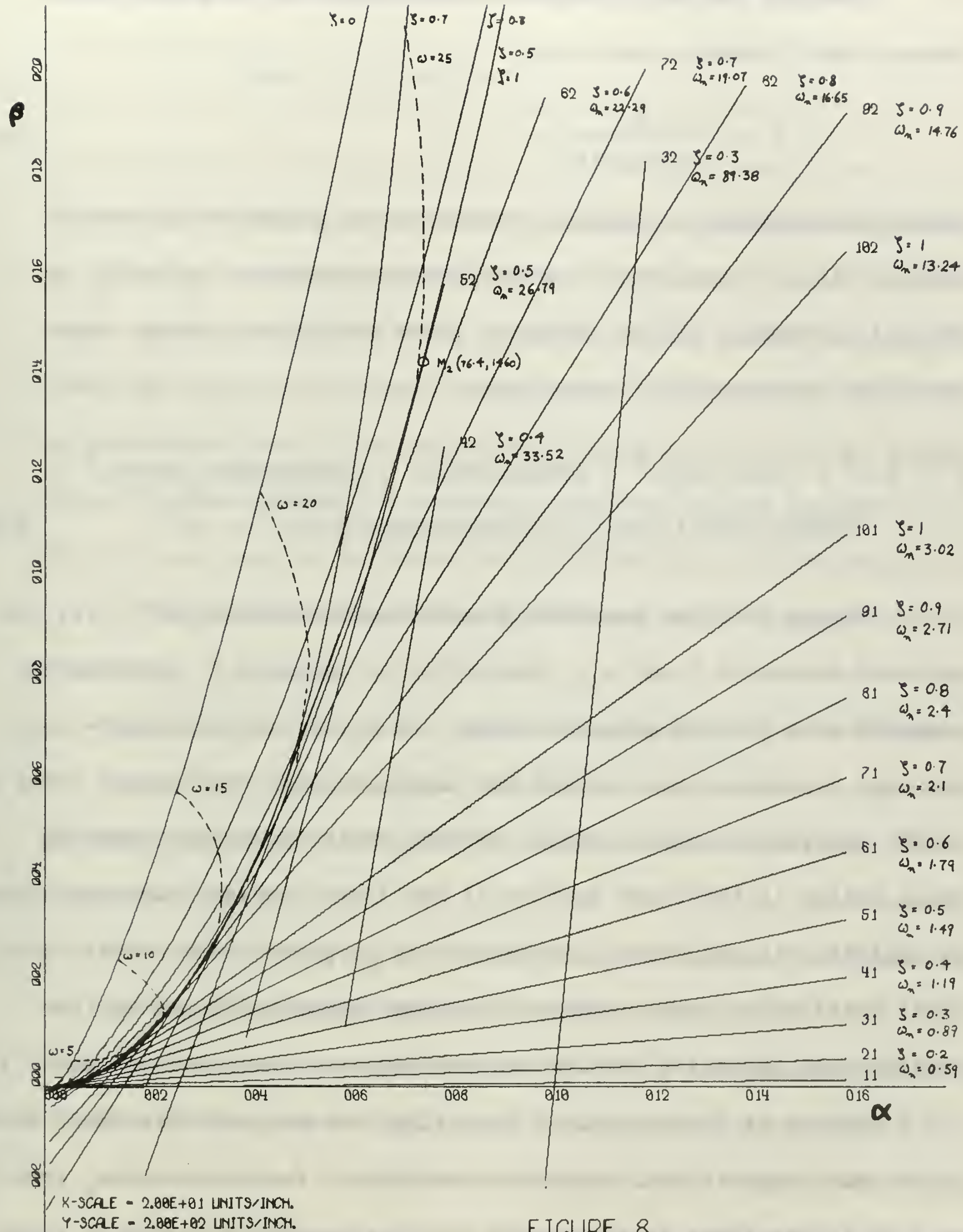


FIGURE 8

SINGULAR LINES ON THE PARAMETER PLANE
OF A MULTIVARIABLE, COUPLED SYSTEM.
6TH. ORDER CHAR. EQUN. K/S2 PLANTS.

Example IX

Consider the basic coupled system structure of Figure 5 but assume plant transfer functions:

$$G = \frac{K}{(s^2+2s+2)} \qquad (8\text{-}11)$$

Assume the compensator transfer functions and parameter values of Example VII. In addition, consider system parameters q_1 and q_2 as defined in Example VII as parameter plane variables α and β. The resulting characteristic equation is

$$s^6 + 84s^5 + (20\alpha+1928)s^4 + (880\alpha+7048)s^3 + (3320\alpha+400\beta+13444)s^2 + (4880\alpha+1600\beta+13120)s + (3200\alpha+1600\beta+1600) = 0 \qquad (8\text{-}12)$$

Figure 9 is the parameter plane diagram of equation (7-12) for selected values of ζ and ω_n. Table VIII of Appendix V, contains the computed data for the singular lines. As ω_n varies from 0 to ∞ the constant ζ contours curl around the origin through the second, third and fourth quadrant of the $\alpha\beta$ plane, reverse their direction producing sharp spikes in the lower section of the first quadrant, and then proceed to infinity via the first quadrant of the $\alpha\beta$ plane. For clarity only the final sections of these contours have been shown in Figure 9; the sections not shown are similar to the spikes of the $\zeta = 1$ contour. The $\zeta = 1$ contour is a particularly confusing contour even with small scale graphs and complete data printouts available. For values of ω_n from 0.2 to 2.06 it describes short arcs in all four quadrants of the $\alpha\beta$ plane with intervening discontinuous jumps. For values of ω_n greater than 2.06 the curve is as shown in Figure 9 crossing the origin and proceeding to infinity via the second quadrant.

The bounds of the stable root area are the $\zeta = 0$ contour and a line determined by setting s=0 in equation (8-12), that is:

$$3200\alpha + \beta + 1600 = 0$$

$$\beta = -(2\alpha + 1) \qquad (8\text{-}13)$$

Within this stable area there are four distinct sections corresponding to different real and complex root combinations. Previous parameter plane methods solved for three pairs of complex roots, two real roots, four real roots, or two real roots and a complex pair in these sections of the stable area. The inclusion of singular lines in the parameter plane diagram defines the remaining roots and allows graphical solution for all six roots of equation (8-12) at any point in the stable area.

The parameter plane diagram and associated singular line data show that:

1. Singular line values are accurately determined for specified values of ζ and ω_n. For example, line 51 for which $\zeta = 0.6$, $\omega_n = 33.272$; the complex roots $s = -19.963 \pm j\,26.618$ are correct to three decimal places.
2. No singular line can be considered as a dominant root line and therefore a dominant root area cannot be defined.
3. The singular lines are technically not coincident with constant σ lines. One spike of the $\zeta = 1$ contour for ω_n from 1.93 to 2.26 produces a mesh of double-valued, constant σ lines which encompasses the stable real root area. The result is that, effectively, each singular line is coincident with a double-valued constant σ line.
4. In part of the stable area singular line intersections occur solving for four roots which were previously undetermined.

5. When a singular line is extended into the fourth quadrant of the $\alpha\beta$ plane below the stable area boundary defined by equation (8-13) the complex roots defined by the singular line remain fixed but the other roots shift to produce a right half s-plane root. For example, since line 73 crosses the stable area boundary as α is decreased from 40.0 to 20.0 a system root moves from $\sigma = -2.02$ to $\sigma = +4.97$.

Example X

Consider the system structure, parameter values, and parameter plane variables of Example VI but now assume plant transfer functions:

$$G = \frac{K}{(s+1)} \qquad (8\text{-}14)$$

The resulting characteristic equation is

$$s^4 + (20\alpha+42)s^3 + (20\alpha+400\beta+161)s^2 + (2440\alpha+1600\beta+3280)s + (1600\alpha+1600\beta+1600) = 0 \qquad (8\text{-}15)$$

The parameter plane diagram for equation (8-14), Figure 10, shows that no singular lines exist. Values of ζ between 0 and 1 were investigated. For all values of ζ less than 1 the coefficient matrix of the parameter plane solution equations, equation (4-4), is non-singular. For ζ=1 there are two values of ω_n for which the coefficient matrix is singular, but for both values the parameter plane solution equations (4-3) are inconsistent.

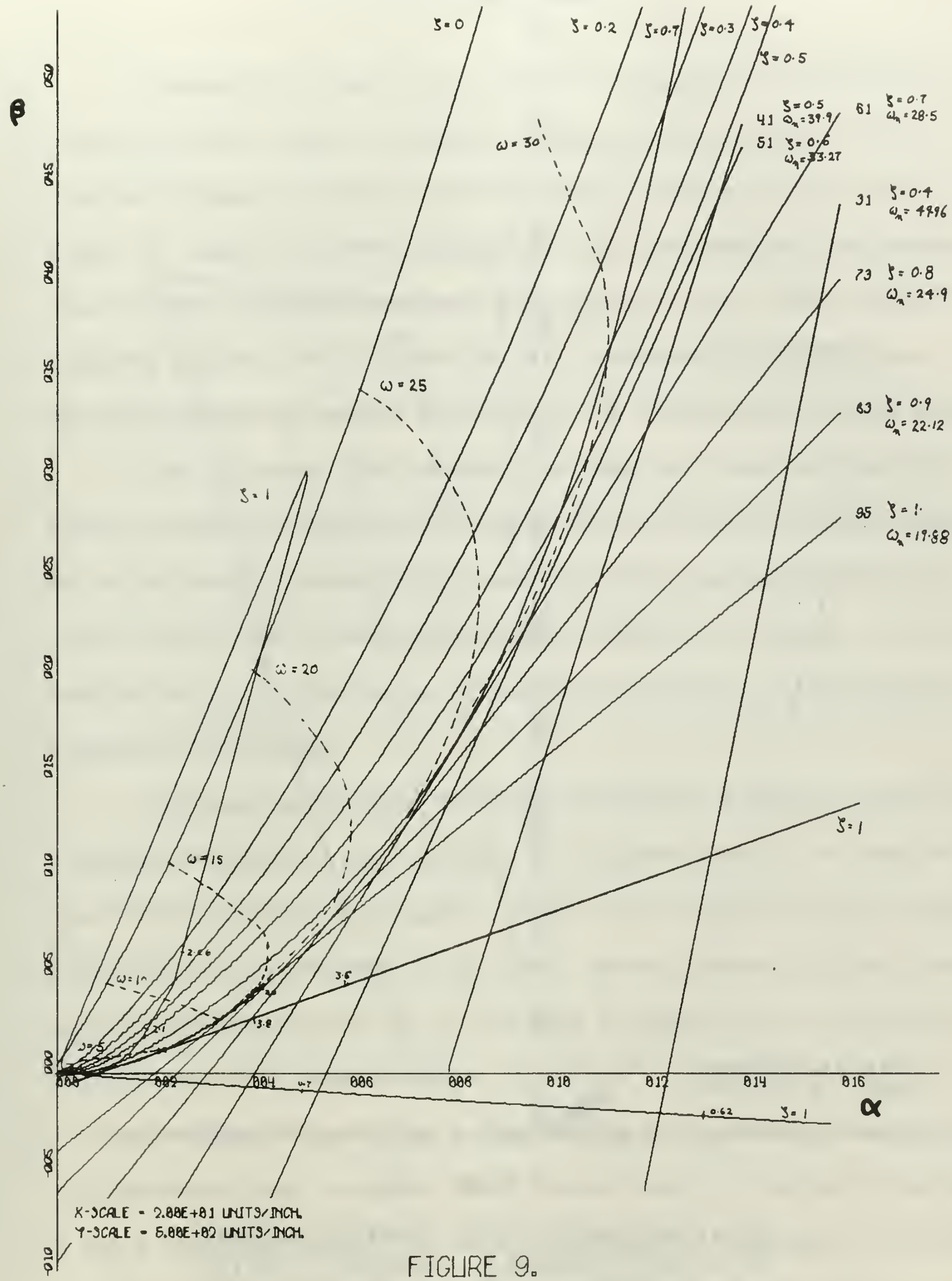


FIGURE 9.

SINGULAR LINES ON THE PARAMETER PLANE
OF A MULTIVARIABLE, COUPLED SYSTEM
6TH. ORDER CHAR.EQUN. K/( ) PLANTS

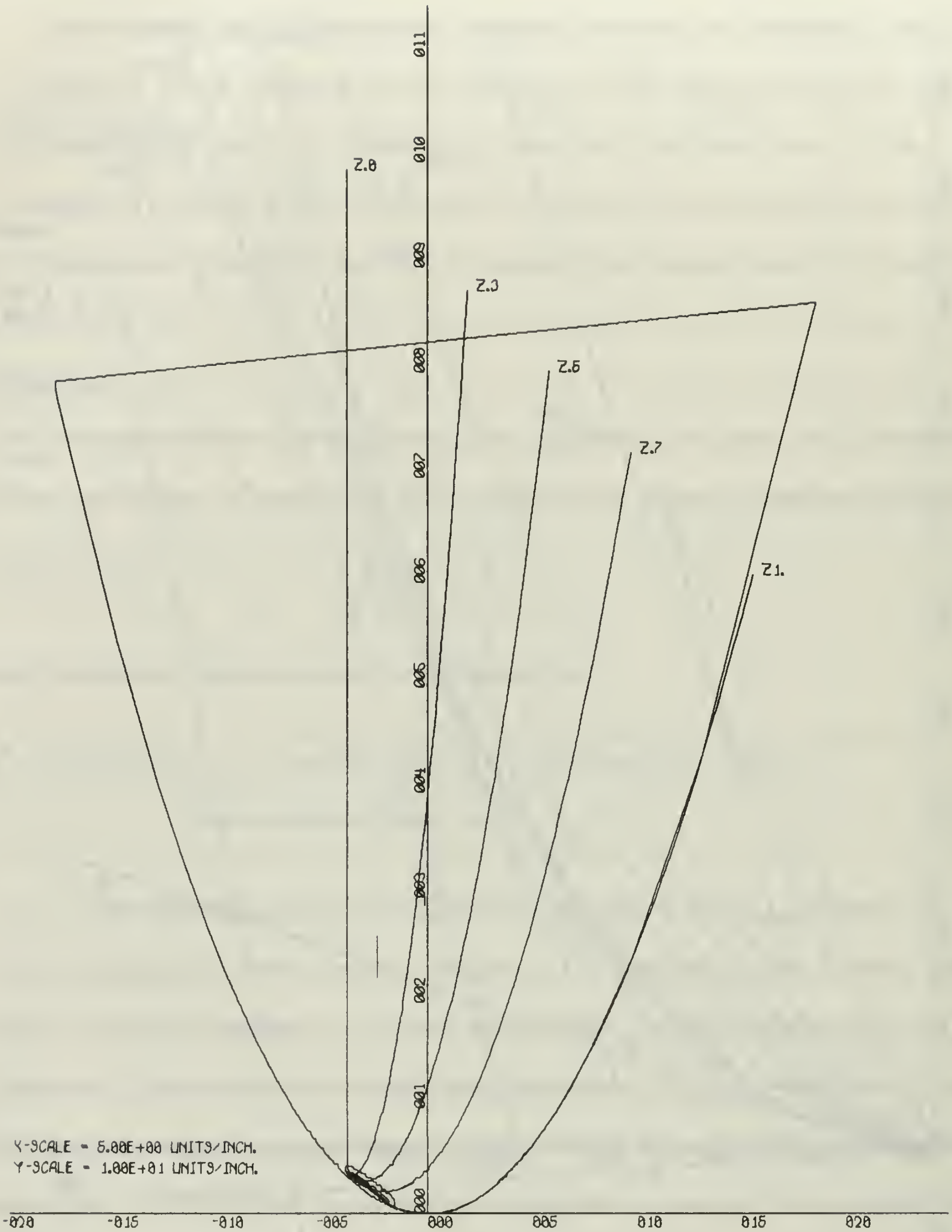


FIGURE 10

SINGULAR LINES ON THE PARAMETER PLANE
OF A MULTIVARIABLE , COUPLED SYSTEM
4TH. ORDER CHAR. EQUN. K/(S+1) PLANTS

IX CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

Singular line theory is a small but meaningful addition to the parameter plane method. Singular lines, constant damping ratio - constant undamped natural frequency lines, occur in only a limited number of cases, but where applicable their addition to the parameter plane diagram defines previously undetermined roots. As a result, complete solution of all characteristic polynomials in which two variable parameters appear linearly in the coefficients is now possible.

The conditions for existence of singular lines are complex. General rules for predicting singular line existence were developed, but an existence theorem which specifies the exact combination of coefficients that allows singular lines should, if possible, be developed. Such an existence theorem would facilitate synthesis and design of singular line systems.

The computer program developed during the course of this study, although relatively slow, provides an accurate means of solving for and graphically displaying singular lines in the parameter plane diagram in conjunction with constant ζ, ω_n, and σ curves. More efficient programming techniques and the use of a faster polynomial root solving subroutine should be considered.

Dominance of singular roots results in a dominant root line on the parameter plane diagram. This allows choice of values of parameters α and β from the infinity of points specified by a singular line while simultaneously ensuring dominant mode operation of the system.

Sensitivities of the damping ratio and undamped natural frequency of singular roots to finite changes in parameters α and β were computed

for specific linear, multivariable control systems. It is considered that measuring the movement of singular roots resulting from finite parameter variations could prove to be a useful concept in self-adaptive control systems. Since movement along a dominant root singular line implies little change in operating conditions, the adaptive controller need only drive the system back to the singular line not to a specific operating point.

Singular line theory is based on the special case of linearly dependent parameter plane solution equations. The theory may not, therefore, be extended to include polynomials whose coefficients are linear functions of two parameters and their product.

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APPENDIX I

TABLE OF THE CHEBYSHEV FUNCTIONS OF THE SECOND KIND

The Chebyshev functions of the second kind, $U_k(\zeta)$, are defined by the recursion relation:

$$U_{k+1}(\zeta) - 2\,U_k(\zeta) + U_{k-1}(\zeta) = 0$$

where:

$$U_0(\zeta) = 0$$

$$U_1(\zeta) = 1$$

These functions for k = -1, ..., 10 are

k	$U_k(\zeta)$
-1	-1
0	0
1	1
2	2ζ
3	$4\zeta^2 - 1$
4	$8\zeta^3 - 4\zeta$
5	$16\zeta^4 - 12\zeta^2 + 1$
6	$32\zeta^5 - 32\zeta^3 + 6\zeta$
7	$64\zeta^6 - 80\zeta^4 + 24\zeta^2 - 1$
8	$128\zeta^7 - 192\zeta^5 + 80\zeta^3 - 8\zeta$
9	$256\zeta^8 - 448\zeta^6 + 240\zeta^4 - 40\zeta^2 + 1$
10	$512\zeta^9 - 1024\zeta^7 + 672\zeta^5 - 160\zeta^3 + 10\zeta$

APPENDIX II

TABLE OF FUNCTIONS $U_k(\zeta)$ [3]

	U_{-1}	U_0	U_1	U_2	U_3	U_4	U_5	U_6	U_7	U_8
0.00				0.0	-1.00	0.000	1.0000	0.00000	-1.000000	0.0000000
0.05				0.1	-0.99	-0.199	0.9701	0.29601	-0.940499	-0.3900599
0.10				0.2	-0.96	-0.392	0.8816	0.56832	-0.767936	-0.7219072
0.15				0.3	-0.91	-0.573	0.7381	0.79443	-0.499771	-0.9443613
0.20				0.4	-0.84	-0.736	0.5456	0.95424	-0.163904	-1.0198016
0.25				0.5	-0.75	-0.875	0.3125	1.03125	0.203125	-0.9296875
0.30				0.6	-0.64	-0.984	0.0496	1.01376	0.558656	-0.6785664
0.35				0.7	-0.51	-1.057	-0.2299	0.89607	0.857149	-0.2960657
0.40				0.8	-0.36	-1.088	-0.5104	0.67968	1.054144	0.1636352
0.45				0.9	-0.19	-1.071	-0.7739	0.37449	1.110941	0.6253569
0.50	-1	0	1	1.0	0.00	-1.000	-1.0000	0.00000	1.000000	1.0000000
0.55				1.1	0.21	-0.869	-1.1659	-0.41439	0.711061	1.1956571
0.60				1.2	0.44	-0.672	-1.2464	-0.82368	0.257984	1.1332608
0.65				1.3	0.69	-0.403	-1.2139	-1.17507	-0.313691	0.7672717
0.70				1.4	0.96	-0.056	-1.0384	-1.39776	-0.918464	0.1119104
0.75				1.5	1.25	0.375	-0.6875	-1.40625	-1.421875	-0.7265625
0.80				1.6	1.56	0.896	-0.1264	-1.09824	-1.630784	-1.5110144
0.85				1.7	1.89	1.513	0.6821	-0.35343	-1.282931	-1.8275527
0.90				1.8	2.24	2.232	1.7776	0.96768	-0.035776	-1.0320768
0.95				1.9	2.61	3.059	3.2021	3.02499	2.545381	1.9112339
1.00				2.0	3.00	4.000	5.0000	6.00000	7.000000	8.0000000
				2	3	4	5			

APPENDIX III

EXPANSION OF $\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$

Given a characteristic polynomial of the form:

$$f(s) = \sum_{k=o}^{n} (b_k \alpha + c_k \beta + d_k)\, s^k \qquad \text{(A-1)}$$

Develop a general expression for:

$$\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = B_1 C_2 - B_2 C_1 \qquad \text{(A-2)}$$

where:

$$B_1 = \sum_{k=o}^{n} (-1)^k b_k\, \omega_n^{\,k} U_{(k-1)} \qquad B_2 = \sum_{k=o}^{n} (-1)^k b_k\, \omega_n^{\,k} U_{(k)} \qquad \text{(A-3)}$$

$$C_1 = \sum_{k=o}^{n} (-1)^k c_k\, \omega_n^{\,k} U_{(k-1)} \qquad C_2 = \sum_{k=o}^{n} (-1)^k c_k\, \omega_n^{\,k} U_{(k)}$$

<u>Consider n 2:</u>

Equation (A-1) becomes:

$$(b_2\alpha + c_2\beta + d_2)s^2 + (b_1\alpha + c_1\beta + d_1)s + (b_o\alpha + c_o\beta + d_o) = 0$$

Evaluating equations (A-3) yields :

$$B_1 = -b_o + b_2\,\omega_n^{\,2} \qquad\qquad B_2 = -b_1\,\omega_n + b_2\,\omega_n^{\,2}(2\zeta)$$

$$C_1 = -c_o + c_2\,\omega_n^{\,2} \qquad\qquad C_2 \quad -c_1\,\omega_n + c_2\,\omega_n^{\,2}(2\zeta)$$

From which:

$$B_1C_2-B_2C_1 \;=\; b_oc_1\,\omega_n - b_2c_1\,\omega_n^{\,3} - b_oc_2\,\omega_n^{\,2}(2\zeta) - b_1c_o\,\omega_n + b_2c_o\,\omega_n^{\,2}(2\zeta) + b_1c_2\,\omega_n^{\,3} \;=\; 0$$

After grouping terms, this is :

$$B_1C_2-B_2C_1 = \left[\begin{array}{l}(b_oc_1-b_1c_o)\,\omega_n - (b_oc_2-b_2c_o)\,\omega_n^{\,2}(2\zeta) \\ +\,(b_1c_2-b_2c_1)\,\omega_n^{\,3} \qquad\qquad 0\end{array}\right] \tag{A-4}$$

<u>Consider n 3 :</u>

Equation (A-1) gives :

$$(b_3\alpha + c_3\beta + d_3)s^3 + (b_2\alpha + c_2\beta + d_2)s^2 + (b_1\alpha + c_1\beta + d_1)s + (b_o\alpha + c_o\beta + d_o) = 0$$

Evaluating equations (A-3) yields :

$$B_1 = -b_o + b_2\,\omega_n^{\,2} - b_3\,\omega_n^{\,3}(2\zeta)$$

$$B_2 = -b_1\,\omega_n + b_2\,\omega_n^{\,2}(2\zeta) - b_3\,\omega_n^{\,3}(4\zeta^2-1)$$

$$C_1 = -c_o + c_2\,\omega_n^{\,2} - c_3\,\omega_n^{\,3}(2\zeta)$$

$$C_2 = -c_1\,\omega_n + c_2\,\omega_n^{\,2}(2\zeta) - c_3\,\omega_n^{\,3}(4\zeta^2-1)$$

Note that in each case only one term of the summation need be calculated, i.e.:

$$B_{1(n)} = B_{1(n-1)} + (-1)^n b_n \omega_n^{\,n} U_{(n-1)} \qquad \text{(A-5)}$$

Multiplying as indicated in equation (A-3), cancelling and grouping terms gives :

$$B_1C_2-B_2C_1 = \begin{bmatrix} (b_oc_1-b_1c_o)\omega_n & -(b_oc_2-b_2c_o)\,\omega_n^{\,2}(2\zeta) & +(b_oc_3-b_3c_o)\omega_n^{\,3}(4\zeta^2-1) \\ +(b_1c_2-b_2c_1)\,\omega_n^{\,3} & -(b_1c_3-b_3c_1)\,\omega_n^{\,4}(2\zeta) & 0 \\ +(b_2c_3-b_3c_2)\,\omega_n^{\,5} & 0 & 0 \end{bmatrix}$$

Note that only the terms involving b_3 and c_3 are new, that is :

$$(B_1C_2-B_2C_1)_{(n)} = (B_1C_2-B_2C_1)_{(n-1)} + \sum_{i=o}^{n-1}(-1)^{i+1}(b_ic_n-b_nc_i)\,{}_n^{\,i-n}U(n-i)$$

<u>Consider n 4 :</u>

Evaluating equations (A-1) in accordance with (A-5) yields :

$$B_1 = B_{1(n=3)} + b_4\,\omega_n^{\,4}(4\zeta^2-1)$$

$$B_2 = B_{2(n=3)} + b_4\,\omega_n^{\,4}(8\zeta^3-4\zeta)$$

$$C_1 = C_{1(n=3)} + c_4\,\omega_n^{\,4}(4\zeta^2-1)$$

$$C_2 = C_{2(n=3)} + c_4\,\omega_n^{\,4}(8\zeta^3-4\zeta)$$

Substituting in equation (A-2), cancelling and grouping terms gives :

$$
B_1C_2 - B_2C_1 = \left[\begin{array}{l}
(b_0c_1 - b_1c_0)\,\omega_n - (b_0c_2 - b_2c_0)\,\omega_n^2(2\zeta) + (b_0c_3 - b_3c_0)\,\omega_n^3(4\zeta^2 - 1) - (b_0c_4 - b_4c_0)\,\omega_n^4(8\zeta^3 - 4\zeta) \\
+ (b_1c_2 - b_2c_1)\,\omega_n^3 - (b_1c_3 - b_3c_1)\,\omega_n^4(2\zeta) + (b_1c_4 - b_4c_1)\,\omega_n^5(4\zeta^2 - 1) \\
+ (b_2c_3 - b_3c_2)\,\omega_n^5 - (b_2c_4 - b_4c_2)\,\omega_n^6(2\zeta) \\
+ (b_3c_4 - b_4c_3)\,\omega_n^7
\end{array}\right]
$$

<u>Extension to the general case :</u>

$$
B_1C_2 - B_2C_1 = \left[\begin{array}{l}
[b_0c_1 - b_1c_0]\,\omega_n + \cdots\cdots + [b_{i-1}c_j - b_j c_{i-1}](-1)^{i-1}U_j(\zeta)\,\omega_n^j + \cdots\cdots + [b_0c_n - b_nc_0](-1)^{n-1}U_n(\zeta)\,\omega_n^n \\
\vdots \qquad\qquad\qquad\qquad \vdots \\
+ [b_{i-1}c_i - b_i c_{i-1}]\,\omega_n^{2i-1} + \cdots\cdots + [b_{i-1}c_{i+j-1} - b_{i+j-1}c_{i-1}](-1)^{i+1}U_j(\zeta)\,\omega_n^{2i+j-2} \\
\vdots \\
+ [b_{n-1}c_n - b_nc_{n-1}]\,\omega_n^{2n-1}
\end{array}\right]
$$

where i 1,2,.....,n is the row number

(A-6)

j 1,2,.....,n is the column number

Or, in compact notation :

$$B_1C_2-B_2C_1 = \sum_{i,j} ce_{ij} \qquad (A-7)$$

where:

$CE = \left[ce_{ij}\right]$ is an (n x m) upper triangular matrix whose elements are :

$$\left[b_{i-1}c_{i+j-1} - b_{i+j-1}c_{i-1}\right] (-1)^{i-1} U_j(\zeta) \quad n^{2i+j-2} \qquad \begin{array}{l} i+j < n-2 \\ i+j \geq n-2 \end{array} \qquad (A-8)$$

APPENDIX IV

PROGRAM PARAM S

```
..JOB0629F, BOWIE D J                                                           000
      PROGRAM PARAM S                                                           001
C  THIS PROGRAM IS APPLICABLE TO POLYNOMIALS WHOSE COEFFICIENTS ARE OF THE
C     FORM  (B*ALPHA + C*BETA + D) WHERE ALPHA AND BETA ARE VARIABLE PARAMETERS
C     AND B, C, AND D ARE CONSTANTS.  THIRD PARAMETERS CAN ALSO BE SPECIFIED
C     AS INDICATED BELOW.
C  THE PROGRAM WILL PLOT ON ONE 9 INCH BY 15 INCH GRAPH PARAMETER PLANE
C     CURVES OF THE FOLLOWING TYPE.
C
C     1. CONSTANT ZETA CURVES AS A FUNCTION OF OMEGA (THE STARTING VALUE OF
             OMEGA AND THE NUMBER OF DECADES THAT OMEGA WILL SPAN WILL BE
             SPECIFIED IN THE DATA CARDS.
      2. CONSTANT OMEGA CURVES FOR PREPROGAMMED VALUES OF ZETA FROM ZERO TO ONE
      3. CONSTANT SIGMA LINES
      4. CONSTANT ZETA - CONSTANT OMEGA CURVES, SINGULAR LINES.
C
C  THE VALUES OF ZETA, OMEGA, SIGMA FOR THE CONSTANT ZETA, CONSTANT OMEGA, AND
C     CONSTANT SIGMA CURVES RESPECTIVELY AND THE VALUES OF ZETA FOR WHICH
C     SINGULAR LINES ARE TO BE COMPUTED MAY BE SPECIFIED IN THE DATA CARDS.
C     IF HOWEVER NO CURVES OF A CERTAIN TYPE ARE DESIRED PLACE A ZERO
C     IN THE APPROPRIATE COLUMN CORRESPONDING TO THE NUMBER OF CURVES. IN
C     THIS CASE SUBMIT A BLANK CARD FOR THE LABELS AND FOR THE CURVE VALUES.
C     IF NO CONSTANT ZETA CURVES ARE DESIRED, SET NZ AND ND TO ZERO, AND
C     SUBMIT A BLANK CARD FOR THE ZETA LABELS, FOR THE ZETA CURVE VALUES, AND
C     FOR THE STARTING VALUE OF OMEGA.
C     ALL CURVES ARE PLOTTED ON THE SAME GRAPH.
C  AN ADDITIONAL FEATURE OF THE PROGRAM IS THAT FAMILIES OF CONSTANT ZETA,
C     SIGMA, AND OMEGA CURVES AND SINGULAR LINES MAY BE PLOTTED IN TERMS OF A
C     THIRD PARAMETER. UP TO 10 VALUES OF THE THIRD PARAMETER MAY BE SPECIFIED.
C     THE THIRD PARAMETER MAY APPEAR LINEARILY OR NON-LINEARILY IN ANY OF THE
C     COEFFICIENTS. THE X-AXIS VARIABLE IS ALPHA AND THE Y-AXIS VARIABLE IS BETA
C     CONSTANT SIGMA LINES WILL BE COMPUTED ONLY FOR THOSE VALUES OF SIGMA ALONG
C     THE NEGATIVE REAL AXIS IN THE S-PLANE. THESE SIGMA VALUES SHOULD BE
C     ENTERED IN THE DATA CARDS AS POSITIVE QUANTITIES.
C  THE VALUE OF ZETA AND OMEGA FOR SINGULAR LINES AND THE ROOTS OF THE INPUT
C     POLYNOMIAL AT SPECIFIED POINTS ON EACH LINE ARE AVAILABLE AS OUTPUT DATA.
```

```
C  THE FOLLOWING SYMBOLS ARE PERTINANT TO THE PROGRAM.
C     ND- THE NUMBER OF DECADES SPANNED BY OMEGA FOR THE CONSTANT ZETA CURVES.
C     NO-THE ORDER OF THE EQUATION.
C     NZ, NS, AND NW- THE NUMBER OF ZETA, SIGMA, AND OMEGA CURVES RESPECTIVELY
C     NZS- THE NUMBER OF VALUES OF ZETA FOR WHICH SINGULAR LINES ARE TO
C     BE COMPUTED
C     IXUP- DISTANCE IN INCHES OF THE X-AXIS FROM THE BOTTOM OF THE GRAPH
C     IYRIGHT-DISTANCE IN INCHES OF THE Y-AXIS FROM THE LEFT SIDE OF THE GRAPH
C     IPRINT- CONTROLS PRINTOUT FOR CONSTANT ZETA,SIGMA,AND OMEGA CURVE DATA
C     ISPRINT- CONTROLS PRINTOUT OF DATA FOR SINGULAR LINES
C     IRPRINT- CONTROLS PRINTOUT OF ROOTS OF INPUT EQUATION AT PREDETERMINED
C     POINTS ON SINGULAR LINES.  (IN EACH CASE ENTER A 1 IF PRINTOUT IS DESIRED.
C     LEAVE BLANK IF PRINTOUT NOT DESIRED.)
C     LABZ,LABS,LABW- THE LABELS FOR THE CONSTANT ZETA, SIGMA, AND OMEGA CURVES
C     LABZS- THE LABELS FOR THE SINGULAR LINES
C     WN- THE STARTING VALUE OF OMEGA FOR THE CONSTANT ZETA CURVES
C     E- THIRD PARAMETER
C     BJ,CJ,DJ- ALPHA, BETA, AND CONSTANT COEFFICIENTS RESPECTIVELY
C  IF A THIRD PARAMETER IS NOT SPECIFIED THE DATA CARDS ARE SUBMITTED IN THE
C     FOLLOWING MANNER.
C     CARD 1 THE FIRST LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 2 THE SECOND LINE OF THE GRAPH TITLE (IN COLUMNS 1-48)
C     CARD 3 IN 8I10 FORMAT ENTER FROM LEFT TO RIGHT
C       ND        NO        NZ        NS        NW        NZS       IXUP    IYRIGHT
C     CARD 4 IN 4I10 FORMAT ENTER FROM LEFT TO RIGHT
C   IPRINT    ISPRINT   IRPRINT       NE
C     CARD 5 LABZ(20A4 FORMAT), LEAVE BLANK IF NZ=0
C     CARD 6 LABS(20A4 FORMAT), LEAVE BLANK CARD IF NS=0
C     CARD 7 LABW(20A4 FORMAT), LEAVE BLANK IF NW=0
C     CARDS 8 LABZS(9A4 FORMAT), ONE CARD FOR EACH VALUE OF ZS.
C               LEAVE A BLANK CARD IF NZS=0.
C     CARD 9 VALUES OF ZETA FOR CONSTANT ZETA CURVES. (8E10.5 FORMAT)
C     CARD 10 VALUES OF SIGMA FOR CONSTANT SIGMA CURVES(8E10.5 FORMAT)
C     CARD 11 VALUES OF OMEGA  FOR CONSTANT OMEGA CURVES(8E10.5 FORMAT)
C     CARD 12 VALUES OF ZETA FOR SINGULAR LINES (8E10.5 FORMAT)
C     CARD 13 CONSTANT COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
```

```
C     CARD 14 ALPHA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 15 BETA COEFFICIENTS IN ASCENDING ORDER (8E10.5 FORMAT)
C     CARD 16 OMIT
C     CARD 17 WN (E10.5 FORMAT)
C     CARD 18 XSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
C     CARD 19 YSCALE (E10.5 FORMAT, USE 1 SIGNIFICANT FIGURE)
      DIMENSION A(350),B(350),ITITLE(12),ZETA(20),LABZ(20,5),SIGMA(20),
     1LABS(20,5),W(20),LABW(20,5),ZS(20),WNS(20,20),LABZS(20,20),
     2AG(350),BG(350),AGS(30),BGS(30),AS(30),BS(30),EJ(20),
     3BJ(20),CJ(20),DJ(20),CE(20,20),CF(20,20),AA(20,20),JSS(20),
     4P(2 ),XAXIS(20,20),SLOPE(20,20),PR(100),PI(100),RR(100),RI(100)
      COMMON E,BJ,CJ,DJ
  250 FORMAT(1H1,17HTHE INPUT DATA IS,////)
 1485 CONTINUE
      PRINT 250
  200 FORMAT (6A8)
      READ 200,(ITITLE(I),I=1,6)
      READ 200,(ITITLE(I),I=7,12)
      PRINT 200,(ITITLE(I),I=1,6)
      PRINT 200,(ITITLE(I),I=7,12)
  251 FORMAT(//,8X,2HND,8X,2HNO,8X,2HNZ,8X,2HNS,8X,2HNW,7X,3HNZS,6X,
     14HIXUP,3X,7HIYRIGHT,//)
      PRINT 251
  203 FORMAT(8I10)
      READ 203,ND,NO,NZ,NS,NW,NZS,IXUP,IYRIGHT
      NC=NO+1
      PRINT 203,ND,NO,NZ,NS,NW,NZS,IXUP,IYRIGHT
  463 FORMAT(//,4X,6HIPRINT,3X,7HISPRINT,3X,7HIRPRINT,8X,2HNE,//)
      PRINT 463
  464 FORMAT (4I10)
      READ 464, IPRINT,ISPRINT,IRPRINT,NE
      PRINT 464, IPRINT,ISPRINT,IRPRINT,NE
      IF(NE) 6789,6789,6790
 6789 NEE=NE+1
      GO TO 6791
 6790 NEE=NE
```

```
6791 AIXUP=IXUP
     AIYRGHT=IYRIGHT
 252 FORMAT(//,4HLABZ,//)
     PRINT 252
 205 FORMAT(20A4)
     READ 205,((LABZ(M,N),M=1,NZ),N=1,NEE)
     PRINT 205,((LABZ(M,N),M=1,NZ),N=1,NEE)
 207 FORMAT(//////,4HLABS,//)
     PRINT 207
     READ 205,((LABS(M,N),M=1,NS),N=1,NEE)
     PRINT 205,((LABS(M,N),M=1,NS),N=1,NEE)
 208 FORMAT(//////,4HLABW,//)
     PRINT 208
     READ 205,((LABW(M,N),M=1,NW),N=1,NEE)
     PRINT 205,((LABW(M,N),M=1,NW),N=1,NEE)
     READ 209,((LABZS(M,N),N=1,9),M=1,NZS)
 209 FORMAT(9A4)
 206 FORMAT(8E10.5)
 210 FORMAT(//////,4HZETA,//)
     PRINT 210
     READ 206,(ZETA(M),M=1,NZ)
     PRINT 206,(ZETA(M),M=1,NZ)
 872 FORMAT(//////,5HSIGMA,//)
     PRINT 872
     READ 206,(SIGMA(M),M=1,NS)
     PRINT 206,(SIGMA(M),M=1,NS)
 212 FORMAT(//////,1HW,//)
     PRINT 212
     READ 206,(W(M),M=1,NW)
     PRINT 206,(W(M),M=1,NW)
 213 FORMAT(//////,2HZS,//)
     PRINT 213
     READ 206,(ZS(M),M=1,NZS)
     PRINT 206,(ZS(M),M=1,NZS)
     IF(NE)214,214,6
 214 FORMAT(//////,37HCONSTANT COEFFICIENTS ASCENDING ORDER,//)
```

```
      PRINT 214
      READ 206,(DJ(N),N=1,NC)
      PRINT 206,(DJ(N),N=1,NC)
  215 FORMAT(//////,34HALPHA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 215
      READ 206,(BJ(N),N=1,NC)
      PRINT 206,(BJ(N),N=1,NC)
  216 FORMAT(//////,33HBETA COEFFICIENTS ASCENDING ORDER,//)
      PRINT 216
      READ 206,(CJ(N),N=1,NC)
      PRINT 206,(CJ(N),N=1,NC)
      GO TO 23
    6 FORMAT(//////,29HVALUES OF THE THIRD PARAMETER,//)
      PRINT 6
      READ 206,(EJ(N),N=1,NE)
      PRINT 206,(EJ(N),N=1,NE)
   23 CONTINUE
  217 FORMAT(//////,22HINITIAL VALUE OF OMEGA,//)
      PRINT 217
  199 FORMAT(E10.5)
      READ 199,WN
      PRINT 199,WN
  218 FORMAT(//////,6HXSCALE,//)
      PRINT 218
      READ 199,XSCALE
      PRINT 199,XSCALE
  418 FORMAT(//////,6HYSCALE,//)
      PRINT 418
      READ 199,YSCALE
      PRINT 199,YSCALE
      ROG=.5+AIYRGHT
      DAV=.5+AIXUP
      AROG=-ROG*XSCALE
      ADAV=-DAV*YSCALE
      ROGE=9.5-AIYRGHT
      DAVE=13.5-AIXUP
```

```
      AROGE=ROGE*XSCALE
      ADAVE=DAVE*YSCALE
      FRAN = -AIYRGHT*XSCALE
      CHEK =-AIXUP*YSCALE
      IF(NZ) 41,41,708
  708 GO TO(61,62,63,64,65,66,67,68,69,7 ),ND
   61 G=1.0076
      GO TO 41
   62 G=1.016
      GO TO 41
   63 G=1.0245
      GO TO 41
   64 G=1.0312
      GO TO 41
   65 G=1.0394
      GO TO 41
   66 G=1.0483
      GO TO 41
   67 G=1.0568
      GO TO 41
   68 G=1.0633
      GO TO 41
   69 G=1.071
      GO TO 41
   70 G=1.078
   41 CONTINUE
      AG(1)=0.0
      BG(1)=0.0
      AG(2)=XSCALE
      BG(2)=0.0
      LABEL=4H
      CALL DRAW(2,AG,BG,1,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
      IF(NZ) 704,704,705
  705 IF(IPRINT)446,446,220
  220 FORMAT(1H1,//////,20HCONSTANT ZETA CURVES,//)
```

```
      PRINT 220
  527 FORMAT(//,15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,16X,4HZETA,5X,
     115HTHIRD PARAMETER,//)
      PRINT 527
  446 MOD=2
      DO 4 ME=1,NEE
      E=EJ(ME)
      IF(NE) 8,8,9
    9 CALL COEF
    8 DO 5 M=1,NZ
      J=0
      JG=
      WNA=WN
      DO 49 L=1,300
      D1= .0
      D2= .0
      C1= .0
      C2= .0
      B1= .0
      B2= .0
      DO 10 N=1,NC
      K=N 1
      IF(K)2,3,2
    3 U=0.0
      U1= 1.0
    2 U2=2.0*ZETA(M)*U-U1
      D1=(-I.0)**K*DJ(N)*WNA**K*U1+D1
      D2=(-1.0)**K*DJ(N)*WNA**K*U+D2
      C1=(-1.0)**K*CJ(N)*WNA**K*U1+C1
      C2=(-1.0)**K*CJ(N)*WNA**K*U+C2
      B1=(-1.0)**K*BJ(N)*WNA**K*U1+B1
      B2=(-1.0)**K*BJ(N)*WNA**K*U+B2
      U1=U
   10 U=U2
      Z=1.0E-60
      IF(ABSF(B1*C2-B2*C1)-Z) 11,11,12
```

```
   11 GO TO 49
   12 J=J 1
      A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
      B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
      IF(IPRINT)447,447,2000
 1001 FORMAT(5E20.5)
 2000 PRINT 1001,A(J),B(J),WNA,ZETA(M),E
  447 IF(FRAN-A(J)) 777,777,49
  777 IF(A(J)-AROGE) 778,778,49
  778 IF(CHEK-B(J)) 779,779,49
  779 IF(B(J)-ADAVE) 800,800,49
  800 JG=JG+1
      AG(JG)=A(J)
      BG(JG)=B(J)
   49 WNA=G*WNA
      CALL DRAW(JG,AG,BG,MOD,0,LABZ(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
    5 CONTINUE
    4 CONTINUE
  704 IF(NS) 22,22,601
  601 IF(IPRINT)448,448,221
  221 FORMAT(1H1,21HCONSTANT SIGMA CURVES,//)
      PRINT 221
  222 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HSIGMA,5X,15HTHIRD PARAMETER,/
     1/)
      PRINT 222
  448 DO 7 ME=1,NEE
      E=EJ(ME)
      IF (NE) 13,13,14
   14 CALL COEF
   13 DO 22 M=1,NS
      DD= .0
      CC= .0
      BB= .0
      DO21 N=1,NC
      K=N 1
```

```
      DD=(-1.0)**K*DJ(N)*SIGMA(M)**K+DD
      CC=(-1.0)**K*CJ(N)*SIGMA(M)**K+CC
      BB=(-1.0)**K*BJ(N)*SIGMA(M)**K+BB
   21 CONTINUE
      J=1
      A(J)=-DD/BB
      B(J)=0.0
      IF(IPRINT)449,449,1002
 1002 FORMAT(4E20.5)
      PRINT 1002,A(J),B(J),SIGMA(M),E
  449 IF(AROG-A(J)) 110,110,310
  110 IF(A(J)-AROGE) 111,111,310
  111 J=J 1
  310 A(J)=0.0
      B(J)=-DD/CC
      IF(IPRINT)450,450,451
  451 PRINT 1002,A(J),B(J),SIGMA(M),E
  450 IF(ADAV-B(J)) 112,112,311
  112 IF(B(J)-ADAVE) 113,113,311
  113 J=J 1
  311 B(J)=(15.0-AIXUP)*YSCALE
      A(J)=(-CC*B(J)-DD)/BB
      IF(IPRINT)452,452,453
  453 PRINT 1002,A(J),B(J),SIGMA(M),E
  452 IF(AROG-A(J)) 114,114,312
  114 IF(A(J)-AROGE) 117,117,312
  312 A(J)=(9.0-AIYRGHT)*XSCALE
      B(J)=(-BB*A(J)-DD)/CC
      IF(IPRINT)454,454,455
  455 PRINT 1002,A(J),B(J),SIGMA(M),E
  454 IF(ADAV-B(J)) 116,116,118
  116 IF(B(J)-ADAVE) 117,117,118
  118 J=J-1
  117 CALL DRAW(J,A,B,2,0,LABS(M,ME),ITITLE,XSCALE,YSCALE,IXUP,IYRIGHT
     1,2,2,9,15,0,LAST)
   22 CONTINUE
```

```
  7 CONTINUE
    IF(NW) 18,18,702
702 IF(IPRINT)459,459,223
223 FORMAT(1H1,21HCONSTANT OMEGA CURVES,//)
    PRINT 223
224 FORMAT(15X,5HALPHA,16X,4HBETA,15X,5HOMEGA,15X,5HAZETA,5X,
   115HTHIRD PARAMETER,//)
    PRINT 224
459 DO 18 ME=1,NEE
    E=EJ(ME)
    IF(NE) 19,19,20
 20 CALL COEF
 19 DO 24 M=1,NW
    J=0
    JG=
    AZETA=0.0
    DO 25 L=1,300
    D1= .0
    D2= .0
    C1= .0
    C2= .0
    B1= .0
    B2= .0
    DO 26 N=1,NC
    K=N 1
    IF(K) 28,27,28
 27 U=0.0
    U1= 1.0
 28 U2=2.0*AZETA*U-U1
    D1=(-1.0)**K*DJ(N)*W(M)**K*U1+D1
    D2=(-1.0)**K*DJ(N)*W(M)**K*U+D2
    C1=(-1.0)**K*CJ(N)*W(M)**K*U1+C1
    C2=(-1.0)**K*CJ(N)*W(M)**K*U+C2
    B1=(-1.0)**K*BJ(N)*W(M)**K*U1+B1
    B2=(-1.0)**K*BJ(N)*W(M)**K*U+B2
    U1=U
```

```
 26 U=U2
    IF(ABSF(B1*C2-B2*C1)-Z)25,25,30
 30 J=J 1
    A(J)=(C1*D2-C2*D1)/(B1*C2-B2*C1)
    B(J)=(B2*D1-B1*D2)/(B1*C2-B2*C1)
    IF(IPRINT)460,460,461
461 PRINT 1001,A(J),B(J),W(M),AZETA,E
460 IF(FRAN-A(J)) 32,32,25
 32 IF(A(J)-AROGE)33,33,25
 33 IF(CHEK-B(J)) 34,34,25
 34 IF(B(J)-ADAVE)35,35,25
 35 JG=JG+1
    AG(JG)=A(J)
    BG(JG)=B(J)
 25 AZETA=AZETA+.00333
 24 CALL DRAW(JG,AG,BG,MOD,0,LABW(M,ME),ITITLE,XSCALE,YSCALE,IXUP,
   1IYRIGHT,2,2,9,15,0,LAST)
 18 CONTINUE
C
C     **********      SINGULAR LINE SUBPROGRAM      **********
C
    IF(NZS)1006,1006,71
 71 IF(ISPRINT)72,72,260
260 FORMAT(1H1, 50HSINGULARITIES. CONSTANT ZETA-CONSTANT OMEGA LINES.
   1,////)
    PRINT 260
 72 DO 107 ME=1,NEE
    E=EJ(ME)
    IF(ME)74,74,73
 73 CALL COEF
C   GENERATE CE MATRIX. ELEMENTS (B(I)C(I+J-B(I+J)C(I)) FOR (I+J) LESS THAN
C   (NO 2),ZERO OTHERWISE.
 74 SUM=0.0
    DO 77 I=1,NO
    DO 77 J=1,NO
    IF((I+J)-(NO+2)) 75,76,76
```

```
   75 CE(I,J)=BJ(I)*CJ(I+J)-BJ(I+J)*CJ(I)
      GO TO 77
   76 CE(I,J)=0.0
   77 SUM = SUM + CE(I,J)
      IF(SUM)79,78,79
   78 PRINT 262
  262 FORMAT(//,10X,61HCOEFFICIENT MATRIX SINGULAR FOR ALL VALUES OF ZET
     1A AND OMEGA,//)
   79 IF(ISPRINT)31,31,263
  263 FORMAT(6E13.5)
C     THIS CARD MUST CONFORM TO ORDER OF EQUATION
      PRINT 263,((CE(I,J),J=1,NO),I=1,NO)
   31 IF(NZS)120,120,80
   80 DO 81 I=1,NO
      DO 81 J=1,NO
   81 CF(I,J)=0.0
      NN=2*NO-1
      DO 89 M=1,NZS
      U=0.0
      U1= 1.0
      DO 82 N=1,NO
      U2=2.0*ZS(M)*U-U1
      CF(N,N)=(-1.)**(N+1)*U2
      U1=U
   82 U=U2
C     GENERATE (AA) MATRIX = (CE)(CF) MATRICES
      DO 83 I=1,NO
      DO 83 J=1,NO
      AA(I,J)=0.0
      DO 83 K=1,NO
   83 AA(I,J)=AA(I,J)+CE(I,K)*CF(K,J)
      DO 86 J =1,NN
      KK=2*NO
      P(J)=0.0
      DO 86 I=1,J
      K=J 2*I+2
```

```
      IF(K)86,86,84
   84 IF(K-NO)85,85,86
   85 P(J)=AA(I,K)+P(J)
   86 CONTINUE
      IU=
      IL=
      DO 52 J =1,NN
      KK=KK-1
      IF(P(J)) 87,50,87
   87 IU=J
   50 IF(P(KK)) 51,52,51
   51 IL=KK
   52 CONTINUE
      IF(IU-IL) 89,89,55
   55 J=0
      DO 53 I =IL,IU
      J=J 1
      PR(J)=P(I)
   53 PI(J)=0.0
      DEL=1.E-20
      JORDER=J-1
      IF(J-2)57,58,59
   57 WNS(M,1)=PR(1)/PR(2)
      GO TO 60
   58 WNS(M,1)=(-PR(2)+SQRTF(PR(2)**2.-4.*PR(1)*PR(3)))/(2.*PR(3))
      WNS(M,2)=(-PR(2)-SQRTF(PR(2)**2.-4.*PR(1)*PR(3)))/(2.*PR(3))
      GO TO 60
   59 CALL POLYRT (PR,PI,JORDER,RR,RI,DEL)
   60 JS=
      JSS(M)=0
      DO 89 I=1,J
      IF(RR(I))109,89,109
  109 IF(ABSF(RI(I))-1.E-5)88,88,89
   88 JS=JS+1
      WNS(M,JS)=RR(I)
      JSS(M)=JS
```

```
 89 CONTINUE
 90 IF(IRPRINT)264,264,91
264 FORMAT(////,13X,5HOMEGA,11X,4HZETA,8X,15HXAXIS INTERCEPT,7X,
   15HSLOPE,6X,15HTHIRD PARAMETER,///)
    PRINT 264
 91 DO 107 M=1,NZS
    IF(JSS(M))108,108,56
 56 JS=JSS(M)
    DO 107 L=1,JS
    B1=.0
    B2=.0
    C1=.0
    C2=.0
    D1=.0
    D2=.0
    DO 94 N=1,NC
    K=N 1
    IF(K)93,92,93
 92 U=0.0
    U1= 1.0
 93 U2=2.0*ZS(M)*U-U1
    X=( 1.0)**K
    WW1=WNS(M,L)**K*U1
    WW=WNS(M,L)**K*U
    D1=D1+X*DJ(N)*WW1
    D2=D2+X*DJ(N)*WW
    C1=C1+X*CJ(N)*WW1
    C2=C2+X*CJ(N)*WW
    B1=B1+X*BJ(N)*WW1
    B2=B2+X*BJ(N)*WW
    U1=U
 94 U=U2
    BC=C2-(B2/B1)*C1
    DB= D2+(B2/B1)*D1
    DC= D2+(C2/C1)*D1
    IF(B1)1,95,1
```

```
 95 IF(B2)1,96,1
 96 IF(C1)1,97,1
 97 IF(C2)1,99,1
  1 IF(ABSF(DB/D2)-1.E-3) 98,98,100
 98 IF(ABSF(DC/D2)-1.E-3) 101,101,100
 99 IF(ISPRINT)107,107,265
265 FORMAT(10X,46HSYSTEM OF EQUATIONS INDETERMINATE.  RANK ZERO. ///)
    PRINT 266, M,L,WNS(M,L),ZS(M)
    PRINT 265
    GO TO 107
100 IF(ISPRINT)107,107,266
266 FORMAT(///,1X,2HWZ,2I2,E13.5,E17.5)
    PRINT 266, M,L,WNS(M,L),ZS(M)
272 FORMAT(10X,47HSYSTEM OF EQUATIONS INCONSISTENT.  NO SOLUTION. ///)
C   GENERATE ALPHA,BETA DEPENDENCE EQUATIONS AND LINE POINTS.
    PRINT 272
    GO TO 107
101 XAXIS(M,L)=-D1/B1
    SLOPE(M,L)=-B1/C1
    IF(IRPRINT)267,267,130
130 PRINT 264
267 FORMAT(1X,2HWZ,2I2,E13.5,4E20.5,//)
    PRINT 267, M,L,WNS(M,L),ZS(M),XAXIS(M,L),SLOPE(M,L),E
    IF(IRPRINT)102,102,268
268 FORMAT(///,20X,5HALPHA,12X,4HBETA,24X,5HROOTS,//)
    PRINT 268
102 JG=
    XK= IYRIGHT
    DO 106 K=1,9
    AS(K)=XK*XSCALE
    BS(K)=(-XAXIS(M,L)+AS(K))*SLOPE(M,L)
    IF(IRPRINT)103,103,269
269 FORMAT (10X,2E17.5,11X,9HREAL PART,7X,14HIMAGINARY PART )
    PRINT 269,AS(K),BS(K)
    DO 54 I=1,NC
 54 PR(I)=BJ(I)*AS(K)+CJ(I)*BS(K)+DJ(I)
```

```
      IF(PR(NC))121,129,121
  129 NO=NO-1
  121 CALL POLYRT (PR,PI,NO,RR,RI,DEL)
      IF(PR(NC))123,122,123
  122 NO=NO+1
      RR(NO)=.0
  123 PRINT 270, (RR(I),RI(I),I=1,NO)
  270 FORMAT (50X,2E20.5)
  103 IF(FRAN-AS(K)) 104,104,106
  104 IF(AS(K)-AROGE)105,105,106
  105 IF(CHEK-BS(K))124,124,106
  124 IF(BS(K)-ADAVE)125,125,106
  125 JG=JG+1
      AGS(JG)=AS(K)
      BGS(JG)=BS(K)
  106 XK=XK+1.
      CALL DRAW(JG,AGS,BGS,MOD,0,LABWZ(M,L),ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,15,0,LAST)
      GO TO 107
  108 IF(M-NZS)107,271,271
  271 FORMAT(///10X,20HMATRIX NONSINGULAR.  )
      PRINT 271
  107 CONTINUE
  120 CONTINUE
C
 1006 AG(1)=0.0
      BG(1)=0.0
      AG(2)=XSCALE
      BG(2)=0.0
      LABEL=4H
      CALL DRAW(2,AG,BG,3,0,LABEL,ITITLE,XSCALE,YSCALE,IXUP,
     1IYRIGHT,2,2,9,13,0,LAST)
      GO TO 1485
      END
      SUBROUTINE COEF
      DIMENSION BJ(100),CJ(100),DJ(100)
```

```
      COMMON E,BJ,CJ,DJ
      COMMON E,BJ,CJ,DJ
      RETURN
      END
```

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APPENDIX V

COMPUTED DATA FOR SINGULAR LINES

TABLE III

COMPUTED DATA FOR SINGULAR LINES OF FIGURE 4

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 1 1	.13062E+01	.20000E+00	-.13419E+01	-.17260E+01	.00000E+00
WZ 1 2	-.10693E+01	.20000E+00	-.13759E+01	-.13717E+01	.00000E+00
WZ 1 3	.11670E+02	.20000E+00	-.19597E+01	.21921E+00	.00000E+00
WZ 1 4	-.10307E+02	.20000E+00	-.25482E+01	-.22704E+00	.00000E+00
WZ 2 1	.13754E+01	.30000E+00	-.13326E+01	-.18893E+01	.00000E+00
WZ 2 2	-.10189E+01	.30000E+00	-.13832E+01	-.13223E+01	.00000E+00
WZ 2 3	.12019E+02	.30000E+00	-.18526E+01	.34788E+00	.00000E+00
WZ 2 4	-.99751E+01	.30000E+00	-.26288E+01	-.32200E+00	.00000E+00
WZ 3 1	-.97160E+00	.40000E+00	-.13901E+01	-.12815E+01	.00000E+00
WZ 3 2	.14490E+01	.40000E+00	-.13231E+01	-.21136E+01	.00000E+00
WZ 3 3	.12370E+02	.40000E+00	-.17260E+01	.48357E+00	.00000E+00
WZ 3 4	-.96471E+01	.40000E+00	-.27124E+01	-.41055E+00	.00000E+00
WZ 4 1	-.92726E+00	.50000E+00	-.13965E+01	-.12474E+01	.00000E+00
WZ 4 2	.15273E+01	.50000E+00	-.13136E+01	-.24361E+01	.00000E+00
WZ 4 3	.12723E+02	.50000E+00	-.15877E+01	.62635E+00	.00000E+00
WZ 4 4	-.93234E+01	.50000E+00	-.27939E+01	-.49283E+00	.00000E+00
WZ 5 1	-.88566E+00	.60000E+00	-.14024E+01	-.12188E+01	.00000E+00
WZ 5 2	.16107E+01	.60000E+00	-.13043E+01	-.29318E+01	.00000E+00
WZ 5 3	.13079E+02	.60000E+00	-.14401E+01	.77629E+00	.00000E+00
WZ 5 4	-.90042E+01	.60000E+00	-.28719E+01	-.56899E+00	.00000E+00
WZ 6 1	-.84664E+00	.70000E+00	-.14078E+01	-.11945E+01	.00000E+00
WZ 6 2	.16993E+01	.70000E+00	-.12953E+01	-.37754E+01	.00000E+00
WZ 6 3	.13437E+02	.70000E+00	-.12841E+01	.93344E+00	.00000E+00
WZ 6 4	-.86900E+01	.70000E+00	-.29456E+01	-.63919E+00	.00000E+00
WZ 7 1	-.81003E+00	.80000E+00	-.14128E+01	-.11737E+01	.00000E+00
WZ 7 2	.17936E+01	.80000E+00	-.12868E+01	-.54919E+01	.00000E+00
WZ 7 3	.13797E+02	.80000E+00	-.11202E+01	.10979E+01	.00000E+00
WZ 7 4	-.83809E+01	.80000E+00	-.30149E+01	-.70359E+00	.00000E+00
WZ 8 1	-.74345E+00	.10000E+01	-.14217E+01	-.11404E+01	.00000E+00
WZ 8 2	.20000E+01	.10000E+01	-.12717E+01	.22785E-304	.00000E+00
WZ 8 3	-.77797E+01	.10000E+01	-.31405E+01	-.81580E+00	.00000E+00
WZ 8 4	.14523E+02	.10000E+01	-.76932E+00	.14488E+01	.00000E+00

TABLE IV

ROOTS OF THE CHARACTERISTIC POLYNOMIAL AT SELECTED POINTS
ON SINGULAR LINE 43 OF FIGURE 4

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 4 3	.12723E+02	.50000E+00	-.15877E+01	.62635E+00	.00000E+00

ALPHA	BETA	ROOTS REAL PART	ROOTS IMAGINARY PART
-.50000E+01	-.21373E+01	-.77018E+00	.15975E-14
		-.63822E+01	.11122E+02
		-.63822E+01	-.11122E+02
		.71535E+02	-.00000E+00
.00000E+00	.99446E+00	-.89742E+00	.00000E+00
		-.64975E+01	.10697E+02
		-.64975E+01	-.10697E+02
		-.28108E+02	.46566E-09
.50000E+01	.41262E+01	-.82323E+00	.24226E-13
		-.63638E+01	.10950E+02
		-.63638E+01	-.10950E+02
		-.12845E+03	-.00000E+00
.10000E+02	.72580E+01	-.81462E+00	.00000E+00
		-.63614E+01	.10981E+02
		-.63614E+01	-.10981E+02
		-.22846E+03	-.00000E+00
.15000E+02	.10390E+02	-.81128E+00	.00000E+00
		-.63611E+01	.10993E+02
		-.63611E+01	-.10993E+02
		-.32847E+03	-.46566E-09

.20000E+02	.13521E+02	REAL PART	IMAGINARY PART
		-.80951E+00	-.71733E-14
		-.63610E+01	.10999E+02
		-.63610E+01	-.10999E+02
		-.42847E+03	-.00000E+00
.25000E+02	.16653E+02	REAL PART	IMAGINARY PART
		-.80841E+00	.00000E+00
		-.63611E+01	.11003E+02
		-.63611E+01	-.11003E+02
		-.52847E+03	-.00000E+00
.30000E+02	.19785E+02	REAL PART	IMAGINARY PART
		-.80766E+00	.00000E+00
		-.63611E+01	.11005E+02
		-.63611E+01	-.11005E+02
		-.62847E+03	-.00000E+00
.35000E+02	.22917E+02	REAL PART	IMAGINARY PART
		-.80712E+00	.00000E+00
		-.63612E+01	.11007E+02
		-.63612E+01	-.11007E+02
		-.72847E+03	-.00000E+00

TABLE V

COMPUTED DATA FOR SINGULAR LINES OF FIGURE 6

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 2 1	.42110E+00	.10000E+00	.17695E+00	.17695E+00	.00000E+00
WZ 2 2	.18998E+03	.10000E+00	.18084E+04	.18084E+04	.00000E+00
WZ 3 1	.84248E+00	.20000E+00	.70380E+00	.70380E+00	.00000E+00
WZ 3 2	.94958E+02	.20000E+00	.45468E+03	.45468E+03	.00000E+00
WZ 4 1	.12644E+01	.30000E+00	.15685E+01	.15685E+01	.00000E+00
WZ 4 2	.63269E+02	.30000E+00	.20402E+03	.20402E+03	.00000E+00
WZ 5 1	.16873E+01	.40000E+00	.27509E+01	.27509E+01	.00000E+00
WZ 5 2	.47413E+02	.40000E+00	.11632E+03	.11632E+03	.00000E+00
WZ 6 1	.21115E+01	.50000E+00	.42229E+01	.42229E+01	.00000E+00
WZ 6 2	.37889E+02	.50000E+00	.75777E+02	.75777E+02	.00000E+00
WZ 7 1	.25373E+01	.60000E+00	.59479E+01	.59479E+01	.00000E+00
WZ 7 2	.31529E+02	.60000E+00	.53800E+02	.53800E+02	.00000E+00
WZ 8 1	.29654E+01	.70000E+00	.78811E+01	.78811E+01	.00000E+00
WZ 8 2	.26977E+02	.70000E+00	.40603E+02	.40603E+02	.00000E+00
WZ 9 1	.33965E+01	.80000E+00	.99690E+01	.99690E+01	.00000E+00
WZ 9 2	.23553E+02	.80000E+00	.32099E+02	.32099E+02	.00000E+00
WZ10 1	.38315E+01	.90000E+00	.12149E+02	.12149E+02	.00000E+00
WZ10 2	.20880E+02	.90000E+00	.26339E+02	.26339E+02	.00000E+00
WZ11 1	.42716E+01	.10000E+01	.14349E+02	.14349E+02	.00000E+00
WZ11 2	.20000E+01	.10000E+01			
	SYSTEM OF EQUATIONS INCONSISTENT. NO SOLUTION.				
WZ11 3	.20000E+01	.10000E+01			
	SYSTEM OF EQUATIONS INCONSISTENT. NO SOLUTION.				
WZ11 4	.18728E+02	.10000E+01	.22301E+02	.22301E+02	.00000E+00

TABLE VI

ROOTS OF THE CHARACTERISTIC POLYNOMIAL AT SELECTED POINTS ON SINGULAR LINE 41 OF FIGURE 6

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE
WZ 4 1	.12644E+01	.30000E+00	.15685E+01	.15685E+01

ALPHA	BETA	ROOTS REAL PART	IMAGINARY PART
.00000E+00	-.24602E+01	-.92798E+00	.13424E-01
		.16602E+01	-.36929E-02
		-.37312E+00	.12014E+01
		-.38539E+00	-.12111E+01
		-.40732E+02	.22692E-04
		-.39241E+02	-.26343E-04
.10000E+02	.13725E+02	REAL PART	IMAGINARY PART
		-.37933E+00	.12062E+01
		-.37933E+00	-.12062E+01
		-.22401E+01	.21159E+01
		-.22401E+01	-.21159E+01
		-.39241E+02	.77449E-09
		-.35520E+02	-.89091E-09
.20000E+02	.28910E+02	REAL PART	IMAGINARY PART
		-.37933E+00	.12062E+01
		-.37933E+00	-.12062E+01
		-.27807E+01	-.79548E-10
		-.95994E+01	.17804E-09
		-.27620E+02	-.36482E-09
		-.39241E+02	.26633E-09
.30000E+02	.44595E+02	REAL PART	IMAGINARY PART
		-.23725E+01	.33634E-14
		-.37933E+00	.12062E+01
		-.37933E+00	-.12062E+01
		-.39241E+02	-.13469E-09
		-.18814E+02	.11198E+02
		-.18814E+02	-.11198E+02

.40000E+02	.60280E+02	REAL PART	IMAGINARY PART
		-.22483E+01	-.10130E-03
		-.37940E+00	.12063E+01
		-.37927E+00	-.12062E+01
		-.39241E+02	.26988E-06
		-.18876E+02	.18096E+02
		-.18876E+02	-.18096E+02
.50000E+02	.75965E+02	REAL PART	IMAGINARY PART
		-.21867E+01	-.12801E-03
		-.37941E+00	.12063E+01
		-.37925E+00	-.12061E+01
		-.39241E+02	.34285E-06
		-.18907E+02	.22989E+02
		-.18907E+02	-.22989E+02
.60000E+02	.91650E+02	REAL PART	IMAGINARY PART
		-.21497E+01	-.13365E-03
		-.37942E+00	.12063E+01
		-.37925E+00	-.12061E+01
		-.39241E+02	.35890E-06
		-.18925E+02	.27002E+02
		-.18925E+02	-.27002E+02
.70000E+02	.10733E+03	REAL PART	IMAGINARY PART
		-.21250E+01	-.13330E-03
		-.37942E+00	.12063E+01
		-.37925E+00	-.12061E+01
		-.39241E+02	.35861E-06
		-.18938E+02	.30488E+02
		-.18938E+02	-.30488E+02
.80000E+02	.12302E+03	REAL PART	IMAGINARY PART
		-.21073E+01	-.13081E-03
		-.37942E+00	.12063E+01
		-.37925E+00	-.12061E+01
		-.39241E+02	.35222E-06
		-.18946E+02	.33613E+02
		-.18946E+02	-.33613E+02

TABLE VII

COMPUTED DATA FOR SINGULAR LINES OF FIGURE 8

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 1 1	.29766E+00	.10000E+00	.88410E-01	.88411E-01	.00000E+00
WZ 1 2	.13432E+03	.10000E+00	.90393E+03	.90395E+03	.00000E+00

NUMPTS MUST NOT BE LESS THAN 2.
THIS PLOT WILL NOT BE OUTPUT.

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 2 1	.59553E+00	.20000E+00	.35164E+00	.35166E+00	.00000E+00
WZ 2 2	.67136E+02	.20000E+00	.22726E+03	.22727E+03	.00000E+00

NUMPTS MUST NOT BE LESS THAN 2.
THIS PLOT WILL NOT BE OUTPUT.

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 3 1	.89385E+00	.30000E+00	.78364E+00	.78375E+00	.00000E+00
WZ 3 2	.44731E+02	.30000E+00	.10197E+03	.10198E+03	.00000E+00
WZ 4 1	.11929E+01	.40000E+00	.13740E+01	.13747E+01	.00000E+00
WZ 4 2	.33521E+02	.40000E+00	.58140E+02	.58146E+02	.00000E+00
WZ 5 1	.14929E+01	.50000E+00	.21153E+01	.21105E+01	.00000E+00
WZ 5 2	.26787E+02	.50000E+00	.37877E+02	.37878E+02	.00000E+00
WZ 6 1	.17942E+01	.60000E+00	.29749E+01	.29730E+01	.00000E+00
WZ 6 2	.22291E+02	.60000E+00	.26896E+02	.26893E+02	.00000E+00
WZ 7 1	.20972E+01	.70000E+00	.39416E+01	.39397E+01	.00000E+00
WZ 7 2	.19073E+02	.70000E+00	.20305E+02	.20296E+02	.00000E+00
WZ 8 1	.24023E+01	.80000E+00	.49862E+01	.49840E+01	.00000E+00
WZ 8 2	.16652E+02	.80000E+00	.16060E+02	.16046E+02	.00000E+00
WZ 9 1	.27104E+01	.90000E+00	.60772E+01	.60747E+01	.00000E+00
WZ 9 2	.14762E+02	.90000E+00	.13188E+02	.13166E+02	.00000E+00
WZ10 1	.30221E+01	.10000E+01	.71787E+01	.71755E+01	.00000E+00
WZ10 2	.13241E+02	.10000E+01	.11175E+02	.11148E+02	.00000E+00

TABLE VIII

COMPUTED DATA FOR SINGULAR LINES OF FIGURE 9

	OMEGA	ZETA	XAXIS INTERCEPT	SLOPE	THIRD PARAMETER
WZ 1 1	.99981E+02	.20000E+00	.49972E+03	.49972E+03	.00000E+00
NUMPTS MUST NOT BE LESS THAN 2. THIS PLOT WILL NOT BE OUTPUT.					
WZ 2 1	.66638E+02	.30000E+00	.22196E+03	.22196E+03	.00000E+00
NUMPTS MUST NOT BE LESS THAN 2. THIS PLOT WILL NOT BE OUTPUT.					
WZ 3 1	.49961E+02	.40000E+00	.12476E+03	.12476E+03	.00000E+00
WZ 4 1	.39950E+02	.50000E+00	.79795E+02	.79795E+02	.00000E+00
WZ 5 1	.33272E+02	.60000E+00	.55391E+02	.55391E+02	.00000E+00
WZ 6 1	.28498E+02	.70000E+00	.40702E+02	.40702E+02	.00000E+00
WZ 7 1	.87286E+00	.80000E+00	.17945E+01	-.12266E+01	.00000E+00
WZ 7 2	.24143E+01	.80000E+00	.54018E+01	.35574E+01	.00000E+00
WZ 7 3	.24913E+02	.80000E+00	.31197E+02	.31196E+02	.00000E+00
WZ 8 1	.70359E+00	.90000E+00	.79286E+00	-.14959E+01	.00000E+00
WZ 8 2	.29985E+01	.90000E+00	.66134E+01	.62273E+01	.00000E+00
WZ 8 3	.22120E+02	.90000E+00	.24713E+02	.24712E+02	.00000E+00
WZ 9 1	.60043E+00	.10000E+01	.38768E+00	-.16323E+01	.00000E+00
WZ 9 2	.20000E+01	.10000E+01	.70789E+01	-.24329E+06	.00000E+00
NUMPTS MUST NOT BE LESS THAN 2. THIS PLOT WILL NOT BE OUTPUT.					
WZ 9 3	.35184E+01	.10000E+01	.89828E+01	.88205E+01	.00000E+00
WZ 9 4	.20000E+01	.10000E+01	.70790E+01	.24330E+06	.00000E+00
NUMPTS MUST NOT BE LESS THAN 2. THIS PLOT WILL NOT BE OUTPUT.					
WZ 9 5	.19881E+02	.10000E+01	.20113E+02	.20112E+02	.00000E+00

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13. ABSTRACT

The theory of Singular Lines, constant damping ratio-constant undamped natural frequency lines, is derived.

A limitation of the parameter plane method for characteristic polynomials whose coefficients are linear functions of two variable parameters which results in undetermined roots is described. The addition of singular lines to the parameter plane diagram specifies these roots allowing solution for all roots of a given polynomial.

A general method of solving for singular lines is developed and rules for predicting the existence of such lines are stated. A computer program which solves for and graphically displays singular lines in addition to constant zeta, omega, and sigma loci is presented.

Singular lines are considered in terms of dominance and macroscopic root sensitivity. A dominant root line in the parameter plane is illustrated.

Examples which demonstrate the application of singular line theory to linear, multivariable control systems are presented.

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